Digital Signal Processing
Discrete-Time Filter Design via Impulse Invariance

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General Requirements for CT/DT Transformation

Requirement 1: Points on the imaginary axis in the s-domain must be mapped to points on the unit circle in the z-domain.

Requirement 2: Stable causal CT transfer functions must be transformed to stable causal DT transfer functions.

Recall that stable causal CT transfer functions have their poles in the left-half plane.
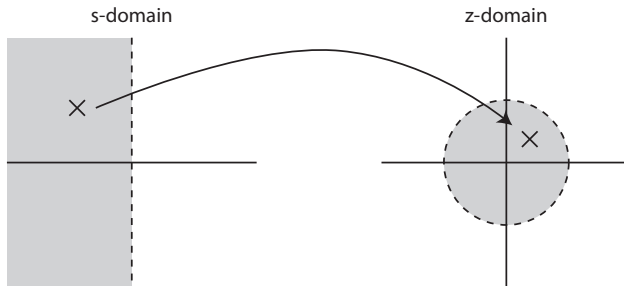
Impulse Invariance Basics

Given a causal LTI CT system impulse response $h_c(t)$, we can set the DT impulse response to match up at the sampling times:

$$h[n] = h_c(t)|_{t=nT} = h_c(nT) \text{ for } n = 0, 1, \ldots$$

Does this CT→DT transformation satisfy the general requirements?

Impulse Invariance: Simple Example

Suppose you are given a causal LTI CT system with $H_c(s) = \frac{1}{s-a}$. The inverse Laplace transform gives $h_c(t) = e^{at}u(t)$. Hence $h[n] = e^{aTn}u[n]$.

What is $H(z)$? Rewrite

$$h[n] = e^{aTn}u[n] = \left(e^{aT}\right)^n u[n] = \alpha^n u[n]$$

Table lookup tells us

$$H(z) = \frac{1}{1 - e^{aT}z^{-1}}$$

with ROC $|z| > e^{aT}$.

Remarks:

- Suppose a is on on the imaginary axis. What is the magnitude of the pole of $H(z)$?
- Suppose a is complex with negative real part. What can you say about the magnitude of the pole of $H(z)$?
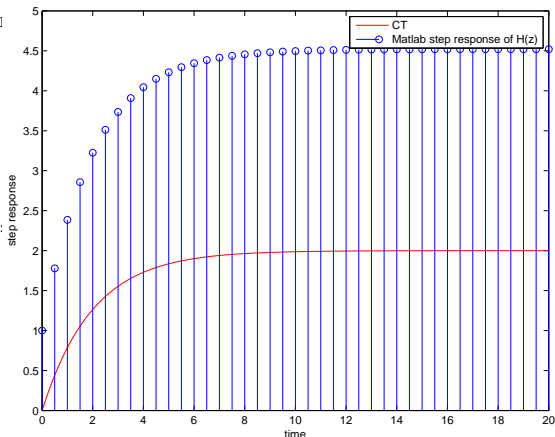
Impulse Invariance Example


```
% CT system
% H(s) = 1/(s-a)
a = -0.5;
tfinal = 20;
ctnum = 1;
ctden = [1 -a];
ctsys = tf(ctnum,ctden);
[y,t] = impulse(ctsys,tfinal);
plot(t,y,'r')

% DT system
T = 0.5;
alpha = exp(a*T);
dtnum = [1 0];
dtden = [1 -alpha];
dtsys = tf(dtnum,dtden,T);
[dty,dtt] = impulse(dtsys,tfinal);
hold on
n=0:length(dtt)-1;
plot(dtt,alpha.^n,'b+');
stem(dtt,dty);
hold off
xlabel('time');
ylabel('impulse response');
legend('CT','h[n]','Matlab impulse response of H(z)');
```
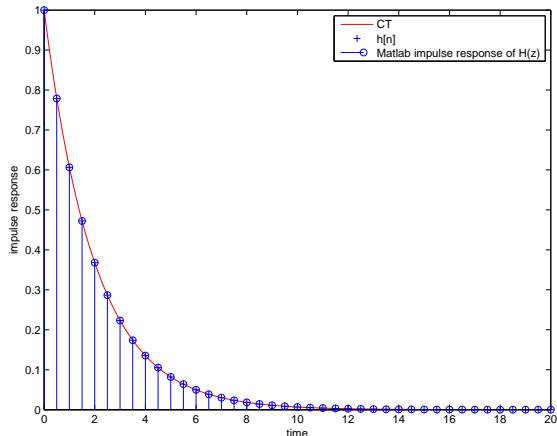
Impulse invariance doesn't imply invariance to all inputs

```
% step responses
% note these results are consistent with
% the final value theorems
[ys,ts] = step(ctsys,tfinal);
[dtys,dtts] = step(dtsys,tfinal);
figure(2)
plot(ts,ys,'r')
hold on
stem(dtts,dtys);
hold off
xlabel('time');
ylabel('step response');
legend('CT','Matlab step response of H(:
```


In general, you can pick a particular input and match the response of the DT system to the response of the CT system for that input. But the response to other inputs will be different.

Impulse Invariance: Frequency Response

Recall that $h[n] = h(nT)$ for $n = 0, 1, \ldots$. What is the relationship between $H(e^{j\omega})$ and $H_c(j\Omega)$?

We've covered the relationship between the CTFT and the DTFT a few times:

$$H(e^{j\omega}) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_c \left(\frac{\omega - k2\pi}{T} \right)$$
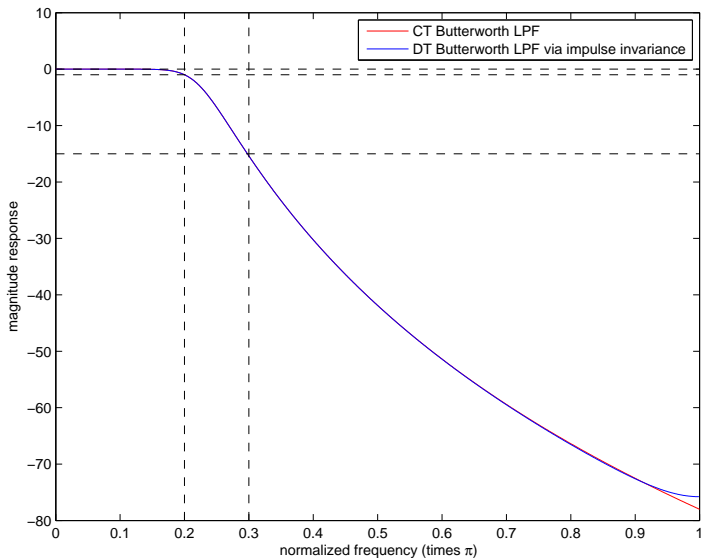
Remarks:

▶ If $H_c(j\Omega)$ is bandlimited such that $H_c(j\Omega) \approx 0$ for all $\Omega \geq \frac{\pi}{T}$, then there will be negligible overlap in the sum of shifted spectra and

$$H(e^{j\omega}) \approx \frac{1}{T} H_c \left(j\frac{\omega}{T} \right) \text{ for } |\omega| \leq \pi$$

Hence, after reconstruction, the DT system designed via impulse invariance accurately emulates the CT system's frequency response.

▶ If $H_c(j\Omega)$ is not bandlimited, e.g. $H_c(j\Omega)$ is a notch filter, there will be aliasing and the resulting DT system's frequency response is not likely to be an accurate emulation of the CT system's frequency response.

Impulse-Invariant Lowpass Butterworth Filter Design Ex.

Impulse Invariance: Final Remarks

1. Main idea is to preserve characteristics of the CT impulse response in the DT impulse response, e.g. fast settling time, etc.

2. Frequency response of DT system is only a good replica of CT system if the CT system is bandlimited (negligible aliasing).

3. Idea can be extended to other types of waveform invariance, e.g. step invariance.

4. The impulse invariance technique maps poles from $s = a_k$ on the s-plane to $z = e^{a_k T}$ on the z-plane.

5. The impulse response of a DT IIR filter is not directly useful for implementation. We need a transfer function or a difference equation.

6. See Matlab function `impinvar`.