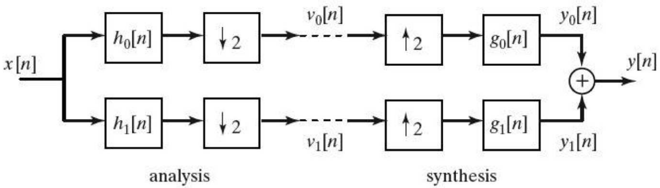
Digital Signal Processing
Multirate Filter Banks

D. Richard Brown III

Multirate Filter Banks and Applications

Example of a two-channel analysis and synthesis multirate filter bank:



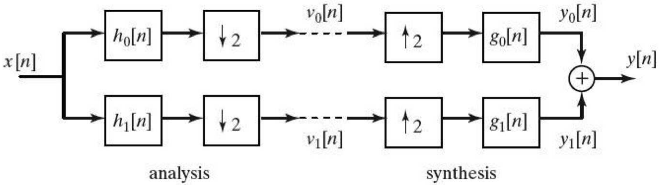
For example: H_0 is a LPF and H_1 is HPF.

Example applications:

- Oversampling in digital audio systems.
- Subband coding of speech and image signals.
- Encryption/security.

See Vaidyanathan, P.P., "Multirate digital filters, filter banks, polyphase networks, and applications: a tutorial," Proceedings of the IEEE , vol.78, no.1, pp.56–93, Jan 1990.

Conditions for Perfect Reconstruction



In the absence of any processing between the down/up-samplers, under what conditions on H_0, H_1, G_0, and G_1 will we have $y[n] = x[n - M]$?

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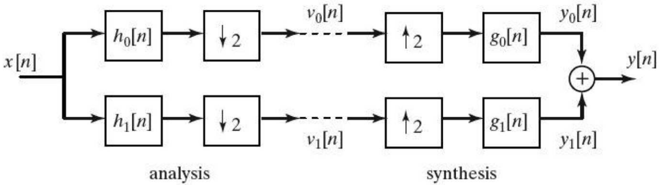
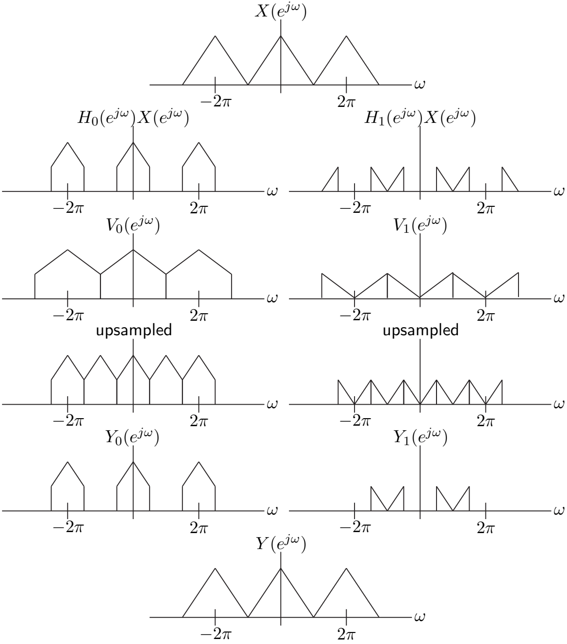
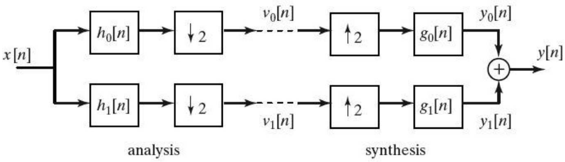


In the absence of any processing between the down/up-samplers, under what conditions on H_0, H_1, G_0, and G_1 will we have $y[n] = x[n-M]$?

Let's see what happens when $H_0 = G_0$ are ideal lowpass filters and $H_1 = G_1$ are ideal highpass filters, all with cutoff frequency $\omega_c = \pi/2$.


$X(e^{j\omega})$
-2π
2π
ω
$H_0(e^{j\omega})X(e^{j\omega})$
$H_1(e^{j\omega})X(e^{j\omega})$
$V_0(e^{j\omega})$
$V_1(e^{j\omega})$
upsampled
upsampled
$Y_0(e^{j\omega})$
$Y_1(e^{j\omega})$
$Y(e^{j\omega})$

Perfect Reconstruction without Ideal Filters



It turns out that we don't need ideal lowpass/highpass filters for this idea to work. As another example, suppose

$$h_0[n] = \frac{1}{2}(\delta[n] + \delta[n-1]) \Leftrightarrow H_0(e^{j\omega}) = \cos(\omega/2)e^{-j\omega/2}$$

$$h_1[n] = \frac{1}{2}(\delta[n] - \delta[n-1]) \Leftrightarrow H_1(e^{j\omega}) = j\sin(\omega/2)e^{-j\omega/2} = H_0(e^{j(\omega-\pi)})$$

$$g_0[n] = \delta[n] + \delta[n-1] \Leftrightarrow G_0(e^{j\omega}) = 2\cos(\omega/2)e^{-j\omega/2} = 2H_0(e^{j\omega})$$

$$g_1[n] = -\delta[n] + \delta[n-1] \Leftrightarrow G_1(e^{j\omega}) = -2j\sin(\omega/2)e^{-j\omega/2} = -2H_0(e^{j(\omega-\pi)})$$

Analysis (part 1 of 2)

Note that the analysis stage filters and downsamples by $M = 2$. Hence, we can write

$$\begin{aligned} V_0(e^{j\omega}) &= \frac{1}{2}\left[X(e^{j\omega/2})H_0(e^{j\omega/2}) + X(e^{j(\omega/2-\pi)})H_0(e^{j(\omega/2-\pi)})\right] \\ &= \frac{e^{-j\omega/4}}{2}\left[X(e^{j\omega/2})\cos(\omega/4) + X(e^{j(\omega/2-\pi)})j\sin(\omega/4)\right] \end{aligned}$$

and

$$\begin{aligned} V_1(e^{j\omega}) &= \frac{1}{2}\left[X(e^{j\omega/2})H_1(e^{j\omega/2}) + X(e^{j(\omega/2-\pi)})H_1(e^{j(\omega/2-\pi)})\right] \\ &= \frac{e^{-j\omega/4}}{2}\left[X(e^{j\omega/2})j\sin(\omega/4) + X(e^{j(\omega/2-\pi)})\cos(\omega/4)\right] \end{aligned}$$

Analysis (part 2 of 2)

The synthesis stage upsamples by $L = 2$ and then filters. Generally, we have

$$Y_i(e^{j\omega}) = G_i(e^{j\omega})V_i(e^{j2\omega})$$

hence we can write

$$\begin{aligned} Y_0(e^{j\omega}) &= 2\cos(\omega/2)e^{-j\omega/2} \cdot \frac{e^{-j\omega/2}}{2}\left[X(e^{j\omega})\cos(\omega/2) + X(e^{j(\omega-2\pi)})j\sin(\omega/2)\right] \\ &= \cos(\omega/2)e^{-j\omega}X(e^{j\omega})\left[\cos(\omega/2) + j\sin(\omega/2)\right] \\ &= e^{-j\omega}X(e^{j\omega})\left[\cos^2(\omega/2) + j\cos(\omega/2)\sin(\omega/2)\right] \end{aligned}$$

and

$$\begin{aligned} Y_1(e^{j\omega}) &= -2j\sin(\omega/2)e^{-j\omega/2} \cdot \frac{e^{-j\omega/2}}{2}\left[X(e^{j\omega})j\sin(\omega/2) + X(e^{j(\omega-2\pi)})\cos(\omega/2)\right] \\ &= -j\sin(\omega/2)e^{-j\omega}X(e^{j\omega})\left[j\sin(\omega/2) + \cos(\omega/2)\right] \\ &= e^{-j\omega}X(e^{j\omega})\left[\sin^2(\omega/2) - j\cos(\omega/2)\sin(\omega/2)\right] \end{aligned}$$

It follows then that

$$Y(e^{j\omega}) = Y_0(e^{j\omega}) + Y_1(e^{j\omega}) = e^{-j\omega}X(e^{j\omega})$$

which means we've recovered our original signal with a one-sample delay.

Remarks

- The filters in the previous examples satisfy the "alias cancellation condition" which requires

$$G_0(e^{j\omega})H_0(e^{j(\omega-\pi)}) + G_1(e^{j\omega})H_1(e^{j(\omega-\pi)}) = 0$$

- You can also use the polyphase implementation ideas we developed earlier to reorder the filtering and up/downsampling in the analysis and synthesis operations to save computation: