Digital Signal Processing
Propagation of Quantization Noise to Filter Output

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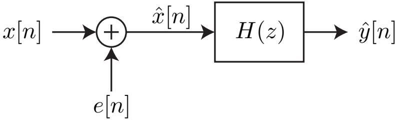
Propagation of Input Quantization Noise to Filter Output

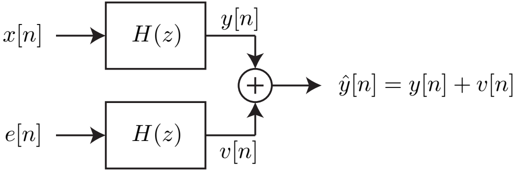
Since the quantized input to $H(z)$ can be written as

$$\hat{x}[n] = x[n] + e[n]$$

we can think of input quantization as shown in below.



Since $H(z)$ is linear, we can use the principle of superposition to analyze the effect of $x[n]$ and $e[n]$ separately. In other words, we can analyze



and look specifically at the properties of $v[n]$.

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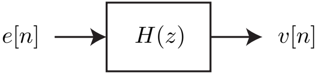
We now focus on analyzing



We assume $H(z)$ is stable. Since $H(z)$ is an LTI system and $\{e[n]\}$ is a zero-mean independent random sequence, i.e. white noise, we can state the following results:

- Output mean:

$$m_v = H(e^{j0})m_e = 0$$

- Output variance:

$$\sigma_v^2 = \sigma_e^2 \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(e^{j\omega})|^2 \, d\omega = \sigma_e^2 \cdot \sum_{n=-\infty}^{\infty} |h[n]|^2$$

Note that $v[n]$ is not a white sequence, in general.

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Example:

$$H(z) = \frac{1}{1 - az^{-1}}$$

with $|a| < 1$ and ROC $|z| > |a|$. We can compute the output variance

$$\begin{aligned}\sigma_v^2 &= \sigma_e^2 \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} |H(\omega)|^2 \, d\omega \\ &= \sigma_e^2 \cdot \frac{1}{2\pi} \int_{-\pi}^{\pi} \left| \frac{1}{1 - ae^{-j\omega}} \right|^2 \, d\omega \\ &= \text{difficult integral}\end{aligned}$$

It is easier to note that $h[n] = a^n u[n]$ and compute

$$\sigma_v^2 = \sigma_e^2 \cdot \sum_{n=-\infty}^{\infty} |h[n]|^2 = \frac{\sigma_e^2}{1 - |\alpha|^2}.$$

What happens when α gets close to the unit circle?

Matlab Simulation of Input Quantization Noise Propagation

```
% output noise variance via simulation
delta = 0.1;            % quantizer step size
a = 0.8;                % filter parameter
num = [1 0]; % numerator coefficients
den = [1 -a]; % denominator coefficients
N = 1e5; % number of samples

% generate input quantization noise sequence
e = rand(1,N)*delta-delta/2;
disp(['Input noise variance           : ' num2str(var(e))]);

% filter
v = filter(num,den,e);

% compute output noise variance
disp(['Output noise variance          : ' num2str(var(v))]);
disp(['Ratio                          : ' num2str(var(v)/var(e))]);
disp(['Analytically predicted ratio : ' num2str(1/(1-abs(a)^2))]);
```

Power Spectrum of Quantization Noise

Since the quantization error is assumed to be white, we have the autocorrelation $\phi_{ee}[m] = \sigma_e^2 \delta[m]$ and the power spectrum

$$\Phi_{ee}(e^{j\omega}) = \sum_{m=-\infty}^{\infty} \phi_{ee}[m] e^{j\omega m} = \sigma_e^2$$

for all ω. The power spectrum of the quantization error at the output of $H(z)$ is then

$$\Phi_{vv}(e^{j\omega}) = |H(e^{j\omega})|^2 \Phi_{ee}(e^{j\omega}) = |H(e^{j\omega})|^2 \sigma_e^2.$$

In fact, after DT→CT reconstruction with $\{v[n]\} \rightarrow e_a(t)$, the CT power spectral density of the quantization error at the output can be written as

$$\Phi_{e_a e_a}(j\Omega) = |H_r(j\Omega) H(e^{j\Omega T})|^2 \sigma_e^2$$

where $H_r(j\Omega)$ is the reconstruction filter (which may not be ideal).