Digital Signal Processing
Oversampled Analog to Digital Conversion with Noise Shaping

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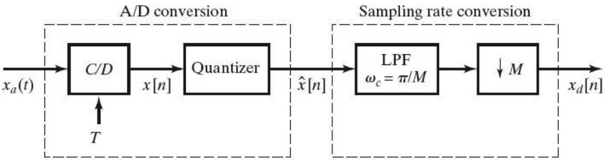
Oversampled ADC System Model

Oversampling with direct quantization model:



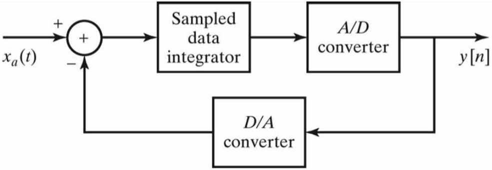
We've previously calculated the SNR of this system as

$$\mathrm{SNR}_Q = 6.02B + 10.8 - 20\log_{10}\left(\frac{X_m}{\sigma_x}\right) + 10\log_{10}(M)$$

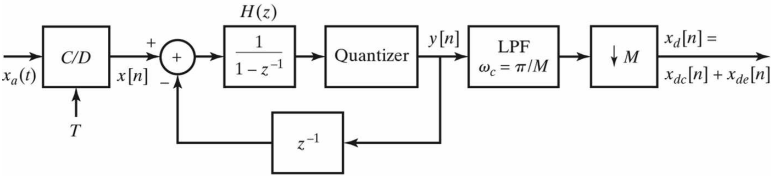
To achieve 60 dB SNR without oversampling, we need $B = 10$ bits. Oversampling by a factor of $M = 256$ reduces the number of necessary bits to $B = 6$.

To achieve more gains, we can combine oversampling with "noise shaping". The main idea is to modify the A/D conversion procedure so that the power spectrum of the quantization noise is pushed outside the passband of the LPF.

Oversampled Quantizer with Noise Shaping System

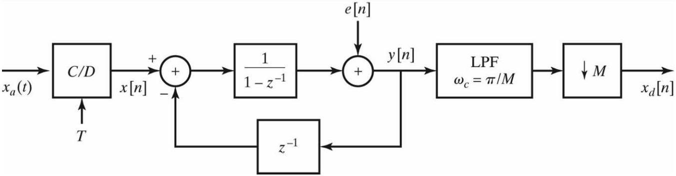


(a)



(b)

Additive Quantization Noise Model (part 1 of 2)



Since everything is linear, we can analyze the transfer function between $x[n]$ and $y[n]$ separately from the transfer function from $e[n]$ to $y[n]$. We have

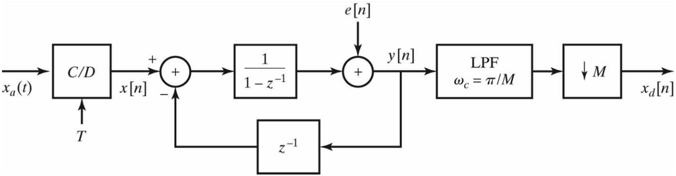
$$Y(z) = \frac{1}{1-z^{-1}}\left(X(z) - z^{-1}Y(z)\right)$$

which can be rearranged as

$$\frac{1}{1-z^{-1}}Y(z) = \frac{1}{1-z^{-1}}X(z)$$

hence $H_x(z) = 1$.

Additive Quantization Noise Model (part 2 of 2)



Looking now at the quantization noise, we have

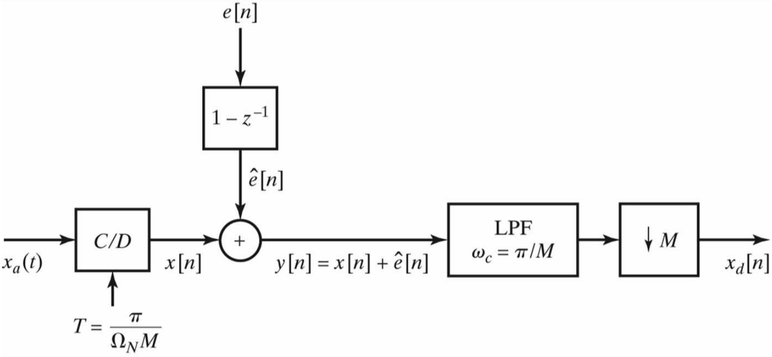
$$Y(z) = \frac{z^{-1}}{1 - z^{-1}} Y(z) + E(z)$$

which can be rearranged as

$$\frac{1}{1 - z^{-1}} Y(z) = X(z)$$

hence $H_e(z) = 1 - z^{-1}$.

Equivalent Model

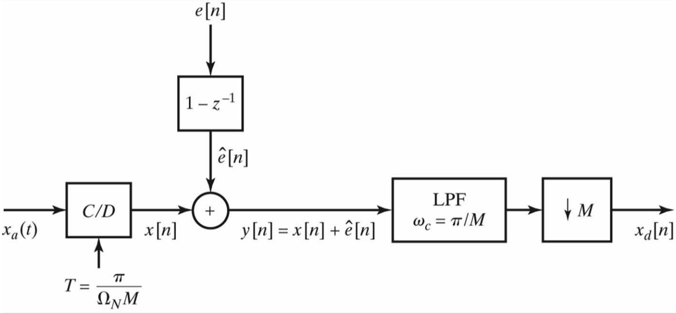


Recall

$$x_d[n] = \underbrace{x_{da}[n]}_{\text{signal part}} + \underbrace{x_{de}[n]}_{\text{noise part}} .$$

We already know from previous analysis that the power of the signal part is unchanged by such a system, i.e., $\sigma^2_{x_{da}} = \sigma^2_x = \sigma^2_{x_a}$.

Noise Power Analysis (part 1 of 2)



Since $\Phi_{ee}(e^{j\omega}) = \sigma_e^2$ for all ω, we have

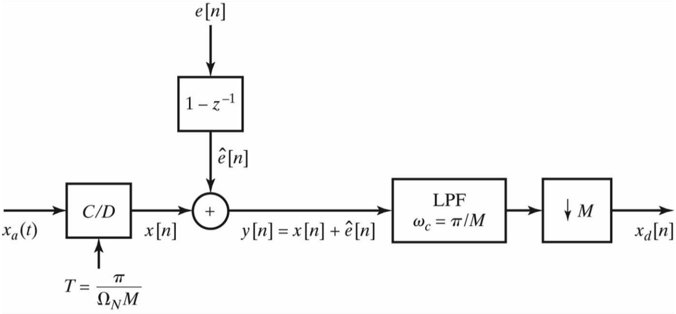
$$\Phi_{\hat{e}\hat{e}}(e^{j\omega}) = |H_e(e^{j\omega})|^2\sigma_e^2 = |1 - e^{-j\omega}|^2\sigma_e^2 = 4\sin^2(\omega/2)\sigma_e^2.$$

Note that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} 4\sin^2(\omega/2)\sigma_e^2\,d\omega = 2\sigma_e^2$$

hence the **total** quantization noise power has doubled after the filter.

Noise Power Analysis (part 2 of 2)

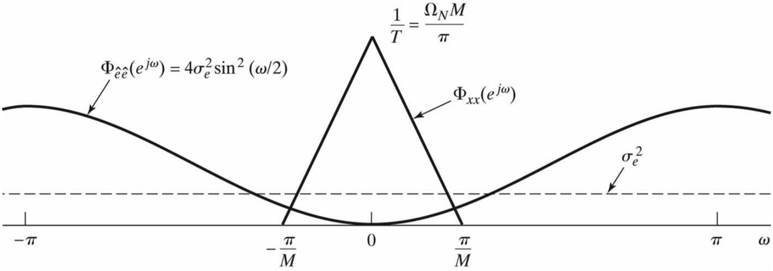


The **actual** quantization noise power that gets through to $x_d[n]$ however is

$$\sigma^2_{x_{de}} = \frac{1}{2\pi}\int_{-\pi/M}^{\pi/M} 4\sin^2(\omega/2)\sigma_e^2\,d\omega = 2\left(\frac{1}{M} - \frac{\sin(\pi/M)}{\pi}\right)\sigma_e^2 \approx 2\left(\frac{\pi^3/M^3}{6\pi}\right)\sigma_e^2$$

where the approximation results from $\sin(x) \approx x - \frac{x^3}{3!}$ for small x.

Oversampled Quantizer with Noise Shaping System



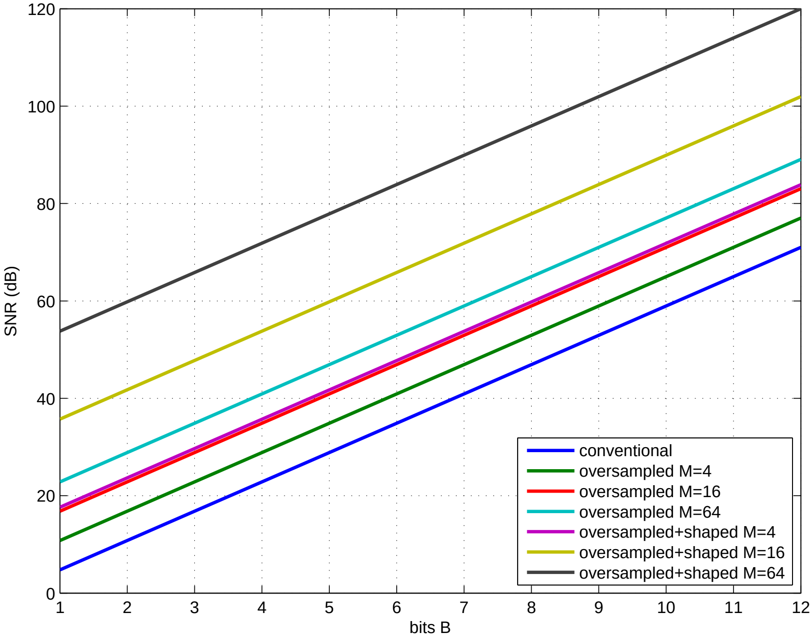
SNR Analysis

Recall that a $B+1$ bit uniform quantizer with step size $\Delta = \frac{X_m}{2^B}$ has $\sigma_e^2 = \frac{\Delta^2}{12} = \frac{X_m^2}{12 \cdot 2^{2B}}$. Hence, the SNR of an oversampled ADC with noise shaping is

$$\begin{aligned}
\mathrm{SNR}_Q &= 10\log_{10}\left(\frac{\sigma_{x_{da}}^2}{\sigma_{x_{de}}^2}\right) \\
&= 10\log_{10}\left(\frac{\sigma_x^2}{2\left(\frac{\pi^2/M^3}{6}\right)\sigma_e^2}\right) \\
&= 10\log_{10}\left(\frac{12 \cdot 2^{2B}\sigma_x^2 \cdot 3M^3}{\pi^2 X_m^2}\right) \\
&= 6.02B + 5.62 - 20\log_{10}\left(\frac{X_m}{\sigma_x}\right) + 30\log_{10}(M)
\end{aligned}$$

Remarks:

- Oversample and shape the quantization noise away from the signal band.
- Doubling M and keeping B fixed gives a 9 dB SNR improvement.
- Quadrupling M and using two less quantizer bits gives the same SNR.


120
100
80
60
40
20
0
SNR (dB)
1
2
3
4
5
6
7
8
9
10
11
12
bits B
conventional
oversampled M=4
oversampled M=16
oversampled M=64
oversampled+shaped M=4
oversampled+shaped M=16
oversampled+shaped M=64