Digital Signal Processing
Phase and Group Delay of LTI Systems

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Recall the frequency response of an LTI system with impulse response $h[n]$ is defined as

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

and represents the complex gain of an LTI system to the eigenfunction input $x[n] = e^{j\omega n}$.

The output of an LTI system with frequency response $H(e^{j\omega})$ and input $x[n] \leftrightarrow X(e^{j\omega})$ is

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}).$$

This last expression can be converted to phase/magnitude (polar) form as

$$|Y(e^{j\omega})| = |H(e^{j\omega})| \cdot |X(e^{j\omega})|$$

$$\angle Y(e^{j\omega}) = \angle H(e^{j\omega}) + \angle X(e^{j\omega})$$
Phase Delay

Previously, we’ve shown that an LTI system \(H(e^{j\omega})\) with input sequence \(x[n] = A \cos(\omega_0 n + \phi)\) for all \(n \in \mathbb{Z}\) yields the output sequence

\[y[n] = |H(e^{j\omega_0})|A \cos (\omega_0 n + \phi + \angle H(e^{j\omega_0}))\]

Denote \(\theta(\omega_0) = \angle H(e^{j\omega_0})\). Then

\[y[n] = |H(e^{j\omega_0})|A \cos (\omega_0 (n + \theta(\omega_0)/\omega_0) + \phi)\]

\[= |H(e^{j\omega_0})|A \cos (\omega_0 (n - \tau_p(\omega_0)) + \phi)\]

where \(\tau_p := -\theta(\omega_0)/\omega_0\) is called the phase delay of the LTI system at frequency \(\omega_0\).

Remarks:

- Note that the units of \(\tau_p(\omega_0)\) are samples.
- Note that \(\tau_p(\omega_0)\) is not necessarily an integer.
- The phase delay \(\tau_p(\omega_0)\) means that the system effectively delays sinusoids at \(\omega_0\) by \(\tau_p(\omega_0)\) samples.
- See Matlab function phasedelay.
Linear Phase Systems

**Definition**

A **linear phase system** is a system with phase response \( \theta(\omega) = \angle H(e^{j\omega}) = -c\omega \) for all \( \omega \) and any constant \( c \).

For example, suppose we have an LTI system with impulse response

\[
h[n] = \{1, 2, 1\}.
\]

We can compute the frequency response

\[
H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = 1 + 2e^{-j\omega} + 1e^{-j2\omega} = (2\cos(\omega) + 2)e^{-j\omega}
\]

We see that \( \theta(\omega) = \angle H(e^{j\omega}) = -\omega \). This is clearly a linear phase system.

Note the **phase delay** of a linear phase system is \( \tau_p(\omega) = -\theta(\omega)/\omega = c \). In other words, all frequencies are delayed by the same amount of time.
Group Delay

Suppose we have an LTI system and a **narrowband** input sequence \( x[n] = A[n]\cos(\omega_0 n + \phi) \). The narrowband assumption means that \( X(\omega) \) is nonzero only around \( \omega = \pm \omega_0 \).

To analyze how an LTI system affects this narrowband signal, we take a Taylor series approximation of the phase response of the LTI system for values of \( \omega \) close to \( \pm \omega_0 \). For values of \( \omega \) close to \( \omega_0 \), we have

\[
\angle H(e^{j\omega}) \approx \theta(\omega_0) + (\omega - \omega_0) \left[ \frac{d\theta(\omega)}{d\omega} \right]_{\omega=\omega_0} = \theta(\omega_0) - (\omega - \omega_0)\tau_g(\omega_0).
\]

Similarly, for values of \( \omega \) close to \( -\omega_0 \), we have

\[
\angle H(e^{j\omega}) \approx \theta(-\omega_0) + (\omega + \omega_0) \left[ \frac{d\theta(\omega)}{d\omega} \right]_{\omega=-\omega_0} = -\theta(\omega_0) - (\omega + \omega_0)\tau_g(-\omega_0).
\]

where

\[
\tau_g(x) := - \left[ \frac{d\theta(\omega)}{d\omega} \right]_{\omega=x}
\]

is called the “group delay” (in samples) at normalized frequency \( x \).
Group Delay

\[ \theta(\omega) = \angle H(e^{j\omega}) \]
Remarks

Suppose you have a narrowband modulated signal \( x[n] = s[n] \cos(\omega_0 n) \) that passes through a system with frequency response \( H(e^{j\omega}) \) with phase delay \( \tau_p(\omega_0) \) and group delay \( \tau_g(\omega_0) \) at \( \omega = \omega_0 \). It can be shown that the output in this case is approximately

\[
y[n] \approx s[n - \tau_g(\omega_0)] \cos(\omega_0(n - \tau_p(\omega_0))).
\]

See Oppenheim & Shafer third edition prob. 5.63 for a detailed derivation.

- Phase delay specifies the delay (in samples) of the “carrier” \( \cos(\omega_0 n) \).
- Group delay specifies the delay (in samples) of the “envelope” \( s[n] \).
- For a linear phase system, \( \tau_g(\omega) = \tau_p(\omega) = c \), i.e. the group delay is the same as the phase delay.
- Group delay is also a measure of the deviation from phase linearity of a system, i.e. if the group delay varies wildly, then the system has highly nonlinear phase.
- See Matlab function `grpdelay`. 