Digital Signal Processing
Phase and Group Delay of LTI Systems

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Review of Basic Concepts

Recall the frequency response of an LTI system with impulse response $h[n]$ is defined as

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n}$$

and represents the complex gain of an LTI system to the eigenfunction input $x[n] = e^{j\omega n}$.

The output of an LTI system with frequency response $H(e^{j\omega})$ and input $x[n] \leftrightarrow X(e^{j\omega})$ is

$$Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}).$$

This last expression can be converted to phase/magnitude (polar) form as

$$\begin{aligned}|Y(e^{j\omega})| &= |H(e^{j\omega})| \cdot |X(e^{j\omega})| \\ \angle Y(e^{j\omega}) &= \angle H(e^{j\omega}) + \angle X(e^{j\omega})\end{aligned}$$

Phase Delay

Previously, we've shown that an LTI system $H(e^{j\omega})$ with input sequence $x[n] = A\cos(\omega_0 n + \phi)$ for all $n \in \mathbb{Z}$ yields the output sequence

$$y[n] = |H(e^{j\omega_0})| A \cos\left(\omega_0 n + \phi + \angle H(e^{j\omega_0})\right)$$

Denote $\theta(\omega_0) = \angle H(e^{j\omega_0})$. Then

$$\begin{aligned} y[n] &= |H(e^{j\omega_0})| A \cos\left(\omega_0 (n + \theta(\omega_0)/\omega_0) + \phi\right) \\ &= |H(e^{j\omega_0})| A \cos\left(\omega_0 (n - \tau_p(\omega_0)) + \phi\right) \end{aligned}$$

where $\tau_p := -\theta(\omega_0)/\omega_0$ is called the **phase delay** of the LTI system at frequency ω_0.

Remarks:

- Note that the units of $\tau_p(\omega_0)$ are **samples**.
- Note that $\tau_p(\omega_0)$ is not necessarily an integer.
- The phase delay $\tau_p(\omega_0)$ means that the system effectively delays sinusoids at ω_0 by $\tau_p(\omega_0)$ samples.
- See Matlab function `phasedelay`.

Linear Phase Systems

Definition

A **linear phase system** is a system with phase response $\theta(\omega) = \angle H(e^{j\omega}) = -c\omega$ for all ω and any constant c.

For example, suppose we have an LTI system with impulse response

$$h[n] = \{1, 2, 1\}.$$

We can compute the frequency response

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = 1 + 2e^{-j\omega} + 1e^{-j2\omega} = (2\cos(\omega) + 2)e^{-j\omega}$$

We see that $\theta(\omega) = \angle H(e^{j\omega}) = -\omega$. This is clearly a linear phase system.

Note the **phase delay** of a linear phase system is $\tau_p(\omega) = -\theta(\omega)/\omega = c$. In other words, all frequencies are delayed by the same amount of time.

Group Delay

Suppose we have an LTI system and a **narrowband** input sequence $x[n] = A[n]\cos(\omega_0 n + \phi)$. The narrowband assumption means that $X(\omega)$ is nonzero only around $\omega = \pm\omega_0$.

To analyze how an LTI system affects this narrowband signal, we take a Taylor series approximation of the phase response of the LTI system for values of ω close to $\pm\omega_0$. For values of ω close to ω_0, we have

$$\angle H(e^{j\omega}) \approx \theta(\omega_0) + (\omega - \omega_0)\left[\frac{d\theta(\omega)}{d\omega}\right]_{\omega=\omega_0} = \theta(\omega_0) - (\omega - \omega_0)\tau_g(\omega_0).$$

Similarly, for values of ω close to $-\omega_0$, we have

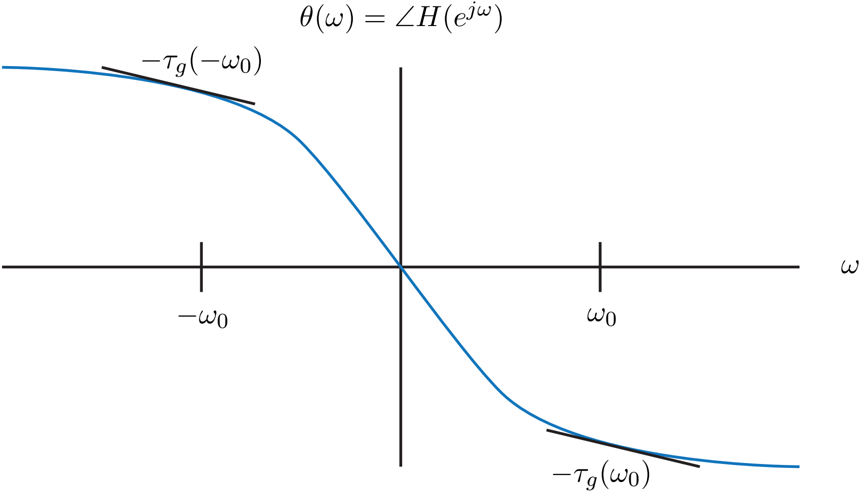
$$\angle H(e^{j\omega}) \approx \theta(-\omega_0) + (\omega + \omega_0)\left[\frac{d\theta(\omega)}{d\omega}\right]_{\omega=-\omega_0} = -\theta(\omega_0) - (\omega + \omega_0)\tau_g(-\omega_0).$$

where

$$\tau_g(x) := -\left[\frac{d\theta(\omega)}{d\omega}\right]_{\omega=x}$$

is called the "group delay" (in samples) at normalized frequency x.

Group Delay


$\theta(\omega) = \angle H(e^{j\omega})$
$-\tau_g(-\omega_0)$
$-\omega_0$
ω_0
ω
$-\tau_g(\omega_0)$

Remarks

Suppose you have a narrowband modulated signal $x[n] = s[n]\cos(\omega_0 n)$ that passes through a system with frequency response $H(e^{j\omega})$ with phase delay $\tau_p(\omega_0)$ and group delay $\tau_g(\omega_0)$ at $\omega = \omega_0$. It can be shown that the output in this case is approximately

$$y[n] \approx s[n - \tau_g(\omega_0)] \cos\left(\omega_0(n - \tau_p(\omega_0))\right).$$

See Oppenheim & Shafer third edition prob. 5.63 for a detailed derivation.

- Phase delay specifies the delay (in samples) of the "carrier" $\cos(\omega_0 n)$.
- Group delay specifies the delay (in samples) of the "envelope" $s[n]$.
- For a linear phase system, $\tau_g(\omega) = \tau_p(\omega) = c$, i.e. the group delay is the same as the phase delay.
- Group delay is also a measure of the deviation from phase linearity of a system, i.e. if the group delay varies wildly, then the system has highly nonlinear phase.
- See Matlab function `grpdelay`.