Digital Signal Processing
Inverse Systems

D. Richard Brown III

Inverse Systems and Equalization

$$x[n] \rightarrow \boxed{H_1(z)} \rightarrow \boxed{H_2(z)} \rightarrow y[n]$$

The system $H_2(z)$ is the inverse of $H_1(z)$ if $y[n] = x[n]$. This is equivalent to saying $H_1(z)H_2(z) = 1$ or $h_1[n] * h_2[n] = \delta[n]$.

Remarks:

1. Note that the zeros of $H_1(z)$ become the poles of $H_2(z)$.
2. Note that the inverse system may not have a unique impulse response unless you further constrain the inverse system to be causal and/or stable (which identifies the ROC).
3. It is often the case that a causal stable inverse can not be found. One workaround is to find a causal stable **generalized inverse** such that $H_1(z)H_2(z) = z^{-n_0}$ for some integer n_0.

Inverse System Region of Convergence

Recall that $H_1(z)$ is associated with a region of convergence $\mathcal{S}_1$.

For $H_1(z)H_2(z) = 1$, there must be a non-empty set $\mathcal{S}$ such that **both** $H_1(z)$ and $H_2(z)$ converge for all $z \in \mathcal{S}$.

Hence, $H_2(z)$ can have any valid ROC $\mathcal{S}_2$ as long as $\mathcal{S}_1 \cap \mathcal{S}_2 \neq \varnothing$.

Example: Suppose

$$H_1(z) = \frac{1 - 0.5z^{-1}}{1 - 0.9z^{-1}}$$

with ROC $\mathcal{S}_1 = \{|z| > 0.9\}$. The inverse system is

$$H_2(z) = \frac{1 - 0.9z^{-1}}{1 - 0.5z^{-1}}$$

with two possible ROCs: $\{|z| < 0.5\}$ or $\{|z| > 0.5\}$. Only one of these ROCs has a non-empty intersection with $\mathcal{S}_1$, however, hence $\mathcal{S}_2 = \{|z| > 0.5\}$. The inverse system is causal and stable.

Another Example

Suppose

$$H_1(z) = 1 - z^{-1}$$

with ROC $\mathcal{S}_1 = \{|z| > 0\}$ since $H_1(z)$ is FIR. The inverse system is

$$H_2(z) = \frac{1}{1 - z^{-1}}$$

with two possible ROCs: $\{|z| < 1\}$ or $\{|z| > 1\}$. Either ROC has a non-empty intersection with $\mathcal{S}_1$, hence

$$h_2[n] = -u[-n-1]$$

for the first ROC or

$$h_2[n] = u[n]$$

for the second ROC. Both yield $h_1[n] * h_2[n] = \delta[n]$.

Some Properties of Inverse Systems

- If $h_1[n]$ is FIR and has two or more coefficients, $h_2[n]$ will be IIR.
- If $H_1(z)$ is an all-pole IIR filter, i.e.,

$$H_1(z) = \frac{b_0}{\prod_i (1 - p_i z^{-1})}$$

  then $H_2(z)$ will be FIR.
- A LTI system $H_1(z)$ is causal and stable and also has a causal and stable inverse if and only if the poles and the zeros of $H_1(z)$ are inside the unit circle. Such systems are called “minimum phase”.

Inverse in Multirate Systems (part 1 of 2)

Consider

$$x[n] \longrightarrow \boxed{\uparrow 2} \longrightarrow \boxed{H_1(z)} \longrightarrow \boxed{H_2(z)} \longrightarrow \boxed{\downarrow 2} \longrightarrow y[n]$$

and suppose $H_1(z)$ is FIR. It is often possible to find FIR $H_2(z)$ so that the overall system from $x[n]$ to $y[n]$ is $z^{-n_0} \leftrightarrow h[n-n_0]$ for some integer n_0.

Example: Suppose $H_1(z) = 1 + 2z^{-1} + 3z^{-2} + 4z^{-3}$ and let $H_2(z) = 1 - 4.5z^{-1} + 6z^{-2}$. Note that both systems are FIR and

$$\bar{H}(z) = H_1(z)H_2(z) = 1 - 2.5z^{-1} - 2.5z^{-3} - 24z^{-5}$$

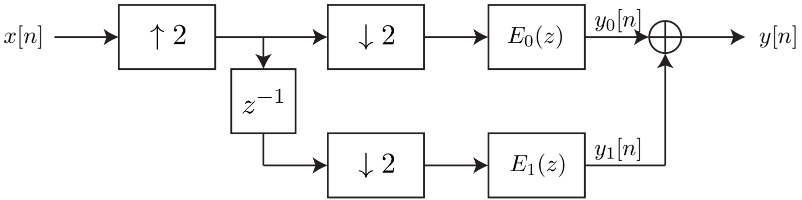
This can be split into an $M = 2$ branch polyphase representation with

$$\begin{aligned} E_0(z) &= 1 \\ E_1(z) &= -2.5 - 2.5z^{-1} - 24z^{-2} \end{aligned}$$

such that $\bar{H}(z) = E_0(z^2) + z^{-1}E_1(z^2)$.

Inverse in Multirate Systems (part 2 of 2)

We can write an equivalent system with a polyphase decimator as



with

$$\begin{aligned} E_0(z) &= 1 \\ E_1(z) &= -2.5 - 2.5z^{-1} - 24z^{-2} \end{aligned}$$

Note that $y_0[n] = x[n]$ and $y_1[n] = 0$. Hence $y[n] = x[n]$ and the overall system from $x[n]$ to $y[n]$ is $H(z) = 1$.