# Digital Signal Processing
# Inferring Frequency Response from Poles and Zeros

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# Rational Transfer Function

Recall, if we have a system with

$$H(z) = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

and the ROC includes the unit circle, then we have

$$H(e^{j\omega}) = \frac{b_0 + b_1 e^{-j\omega} + \cdots + b_M e^{-j\omega M}}{a_0 + a_1 e^{-j\omega} + \cdots + a_N e^{-j\omega N}} \tag{1}$$

which can be factored to write

$$H(e^{j\omega}) = \left(\frac{b_0}{a_0}\right) \frac{(1 - c_1 e^{-j\omega}) \cdots (1 - c_M e^{-j\omega})}{(1 - d_1 e^{-j\omega}) \cdots (1 - d_N e^{-j\omega})} \tag{2}$$

The MATLAB commands `roots` and `poly` allow easy conversion between (1) and (2).

# Magnitude Response

We can write the magnitude response as

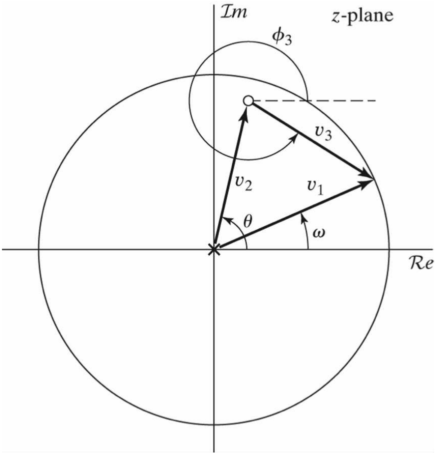
$$|H(e^{j\omega})| = \left|\frac{b_0}{a_0}\right| \frac{|1 - c_1 e^{-j\omega}| \cdots |1 - c_M e^{-j\omega}|}{|1 - d_1 e^{-j\omega}| \cdots |1 - d_N e^{-j\omega}|}$$
$$= \left|\frac{b_0}{a_0}\right| \frac{|e^{j\omega} - c_1| \cdots |e^{j\omega} - c_M|}{|e^{j\omega} - d_1| \cdots |e^{j\omega} - d_N|}$$

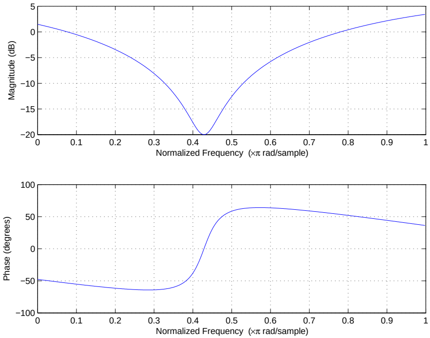
Intuition:

- $e^{j\omega}$ is a point on the unit circle.
- $|e^{j\omega} - c_m|$ is the distance from the $m^{\text{th}}$ zero to that point on the unit circle.
- $|e^{j\omega} - d_n|$ is the distance from the $n^{\text{th}}$ pole to that point on the unit circle.
- $|H(e^{j\omega})|$ is large when $e^{j\omega}$ is close to a pole and far from the zeros.
- $|H(e^{j\omega})|$ is small when $e^{j\omega}$ is close to a zero and far from the poles.

# 1st Order FIR Filter Example: $H(z) = 1 - az^{-1}$

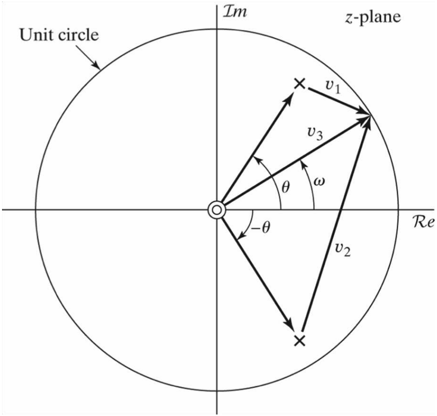
$a = 0.9e^{j3\pi/7}$

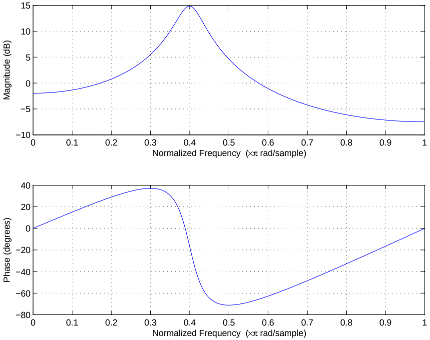


# 2nd Order IIR Filter Example: $H(z) = \frac{1}{1-2r\cos(\theta)z^{-1}+r^2z^{-2}}$

$r = 0.9$ and $\theta = 2\pi/5$



# Phase Response

We can write the phase response as

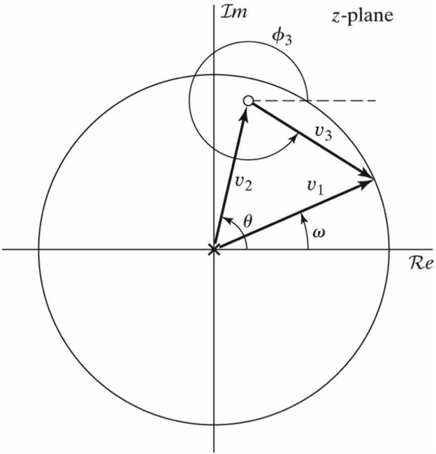
$$\begin{aligned}\angle H(e^{j\omega}) &= \angle\frac{b_0}{a_0} + \sum_{m=1}^{M} \angle(1 - c_m e^{-j\omega}) - \sum_{n=1}^{N} \angle(1 - d_n e^{-j\omega}) \\ &= \angle\frac{b_0}{a_0} + \sum_{m=1}^{M} \left[\angle(e^{j\omega} - c_m) - \angle e^{j\omega}\right] - \sum_{n=1}^{N} \left[\angle(e^{j\omega} - d_n) - \angle e^{j\omega}\right]\end{aligned}$$

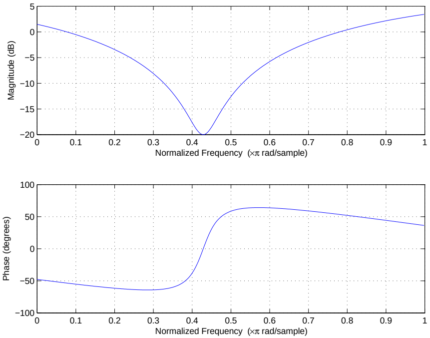
Intuition:

- $e^{j\omega}$ is a point on the unit circle.
- $\angle(e^{j\omega} - c_m)$ is the angle of the vector from the $m^{\text{th}}$ zero to that point on the unit circle.
- $\angle(e^{j\omega} - d_n)$ is the angle of the vector from the $n^{\text{th}}$ pole to that point on the unit circle.
- $\angle e^{j\omega} = \omega$.

# 1st Order FIR Filter Example: $H(z) = 1 - az^{-1}$

$a = 0.9e^{j3\pi/7}$



# 2nd Order IIR Filter Example: $H(z) = \frac{1}{1-2r\cos(\theta)z^{-1}+r^2z^{-2}}$

$r = 0.9$ and $\theta = 2\pi/5$