# Digital Signal Processing
# All-Pass Systems

D. Richard Brown III

# All-Pass System Basics

## Definition

A stable LTI system $\mathcal{H}$ with rational transfer function $H(z)$ is called **all-pass** if its DTFT satisfies $|H(e^{j\omega})| = A$ for some value $A > 0$ and all $\omega$.

All-pass systems only affect the phase of the signal. They are often used to compensate for undesired phase shifts or delays in other systems without affecting the shape of the magnitude response.

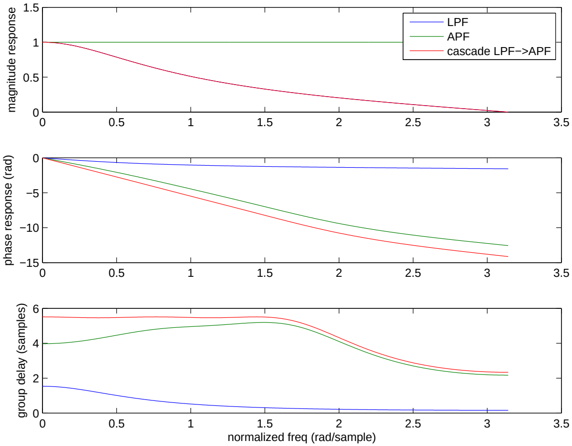
Example: The delay system $H(z) = z^{-n_0}$ is clearly an all-pass system since $H(e^{j\omega}) = e^{-j\omega n_0}$ and $|H(e^{j\omega})| = 1$.

Remarks:

- All-pass systems typically do not have linear phase.
- Some textbooks define all-pass systems such that $|H(e^{j\omega})| = 1$.

# All-pass Delay Equalizer Example

It is not possible to design a stable causal IIR filter with linear phase. So what we can do instead is cascade an all-pass filter with an IIR filter to get approximately linear phase over a desired range of frequencies. Example:



# Properties of All-Pass Systems

Since we require $|H(e^{j\omega})| = A$ for some value $A > 0$ and all $\omega$, it is equivalent to write

$$
\begin{aligned}
& |H(e^{j\omega})|^2 = A^2 \\
\Leftrightarrow \quad & H(e^{j\omega})H^*(e^{j\omega}) = A^2 \\
\Leftrightarrow \quad & h[n] * h^*[-n] = A^2\delta[n] \\
\Leftrightarrow \quad & H(z)H^*(1/z^*) = A^2
\end{aligned}
$$

Interpretation: $H(z)$ must have the property that $H^*(1/z^*)$ is its inverse system (to a scale factor). This means that $H(z)$ is all-pass if and only if

- the poles of $H(z)$ are canceled by the zeros of $H^*(1/z^*)$ and
- the zeros of $H(z)$ are canceled by the poles of $H^*(1/z^*)$.

# First-Order All-Pass System

As an example, suppose

$$F(z) = 1 - az^{-1}.$$

Then

$$F^*(1/z^*) = (1 - a(1/z^*)^{-1})^* = 1 - a^*z.$$

Hence, we can form a first-order all-pass filter as

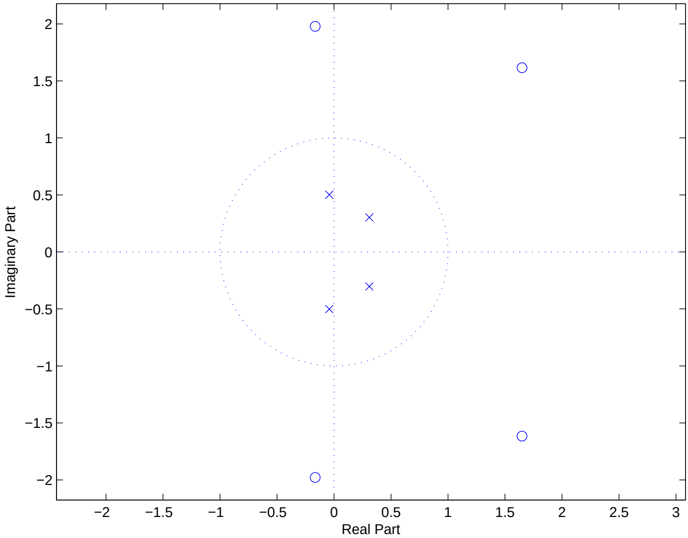
$$H(z) = \frac{F^*(1/z^*)}{F(z)} = \frac{1 - a^*z}{1 - az^{-1}}$$

Since $H^*(1/z^*) = \frac{F(z)}{F^*(1/z^*)}$ it should be clear that $H(z)H^*(1/z^*) = 1$.

Remarks:

- We have poles at $z = a$ and $z = 0$.
- We have zeros at $z = \frac{1}{a^*}$ and $z = \infty$.
- In general, all-pass filters have conjugate-reciprocal pole-zero pairs.

# All-pass Filter Poles and Zeros Mirrored Across Unit Circle



# General All-Pass Systems

Given a first-order all-pass filter

$$H(z) = \frac{F^*(1/z^*)}{F(z)} = \frac{1 - a^* z}{1 - az^{-1}}$$

note that $cz^{-n_0} H(z)$ is also all-pass if $c > 0$.

Also observe that the cascade of two or more all-pass filters is all-pass.

Hence, transfer functions of the form

$$H(z) = cz^{-n_0} \prod_i \frac{1 - a_i^* z}{1 - a_i z^{-1}}$$

where each pole is paired with a conjugate reciprocal zero are all-pass. The phase response follows as

$$\angle H(e^{j\omega}) = \angle c + \angle e^{-j\omega n_0} + \sum_i \angle(1 - a_i^* e^{j\omega}) - \sum_i \left[\angle(e^{j\omega} - a_i) - \angle e^{j\omega}\right]$$

See Matlab functions `isallpass` and `iirgrpdelay`.