Digital Signal Processing
Minimum-Phase Systems

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Phase Response Characterization of Transfer Function

Definition

A causal stable LTI system $\mathcal{H}$ with transfer function $H(z)$ with all zeros inside the unit circle is called **minimum phase**.

Definition

A causal stable system $\mathcal{H}$ with transfer function $H(z)$ with all zeros outside the unit circle is called **maximum phase**.

Definition

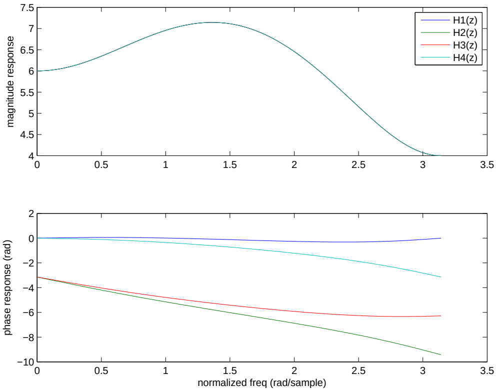
A causal stable system $\mathcal{H}$ with transfer function $H(z)$ with at least one zero inside the unit circle and at least one zero outside the unit circle is called **mixed phase**.

Minimum phase systems are important because they have a stable inverse $G(z) = 1/H(z)$. You can convert between min/max/mixed-phase systems by cascading allpass filters.

Minimum Phase Systems

When we say a system is "minimum phase", we mean that it has the least phase delay (or least phase lag) among all systems with the same magnitude response.

$$
\begin{aligned}
H_1(z) &= 6 + z^{-1} - z^{-2} \\
&= 6(1 + z^{-1}/2)(1 - z^{-1}/3) \\
H_2(z) &= 1 - z^{-1} - 6z^{-2} \\
&= (1 + 2z^{-1})(1 - 3z^{-1}) \\
H_3(z) &= 2 - 5z^{-1} - 3z^{-2} \\
&= 2(1 + z^{-1}/2)(1 - 3z^{-1}) \\
H_4(z) &= 3 + 5z^{-1} - 2z^{-2} \\
&= (1 + 2z^{-1})(1 - z^{-1}/3)
\end{aligned}
$$



Minimum phase systems with real-valued impulse responses have $\angle H(e^{j0}) = 0$.

Phase Analysis

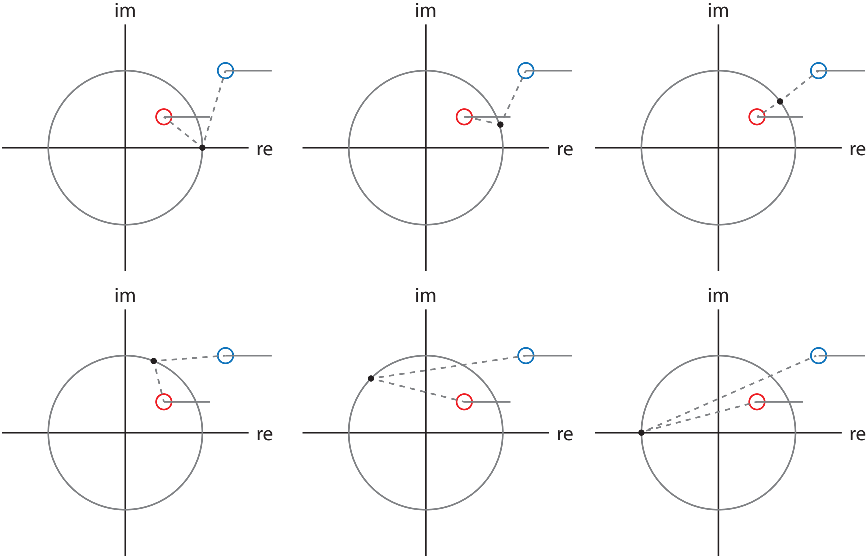
Recall that the phase response of a rational $H(z)$ can be written as

$$\begin{aligned}\angle H(e^{j\omega}) &= \angle\frac{b_0}{a_0} + \sum_{m=1}^{M}\angle(1-c_m e^{-j\omega}) - \sum_{n=1}^{N}\angle(1-d_n e^{-j\omega}) \\ &= \angle\frac{b_0}{a_0} + \sum_{m=1}^{M}\left[\angle(e^{j\omega}-c_m) - \angle e^{j\omega}\right] - \sum_{n=1}^{N}\left[\angle(e^{j\omega}-d_n) - \angle e^{j\omega}\right]\end{aligned}$$

To have the least phase delay for a causal system, we must have

$$\angle(e^{j\omega}-c_m) > \angle(e^{j\omega}-1/c_m^*)$$

for all $|c_m| < 1$. We will illustrate this graphically...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Minimum Phase $\Rightarrow$ Minimum Group Delay

Suppose we have a rational transfer function with the term $(1 - cz^{-1})$ in the numerator with $c = ae^{jb}$ and $0 < a < 1$. We can write

$$\begin{aligned}\angle(1 - ce^{-j\omega}) &= \angle(1 - ae^{-j(\omega-b)}) \\ &= \angle\left(1 - a\cos(\omega - b) + aj\sin(\omega - b)\right) \\ &= \arctan\left(\frac{a\sin(\omega - b)}{1 - a\cos(\omega - b)}\right)\end{aligned}$$

Note that

$$\frac{\partial}{\partial\omega}\arctan\left(\frac{f(\omega)}{g(\omega)}\right) = \frac{f'(\omega)g(\omega) - g'(\omega)f(\omega)}{g^2(\omega) + f^2(\omega)}$$

Omitting the algebraic details, the group delay then follows as

$$\tau_g(\omega) = -\frac{\partial}{\partial\omega}\angle(1 - ce^{-j\omega}) = \frac{a^2 - a\cos(\omega - b)}{1 + a^2 - 2a\cos(\omega - b)} = \frac{a - \cos(\omega - b)}{a^{-1} + a - 2\cos(\omega - b)}$$

Note that the denominator is unaffected by replacing a with $a^{-1} > 1$. But the numerator gets larger if we replace a with $a^{-1} > 1$.

Minimum Phase $\Rightarrow$ Minimum Energy Delay

Given a causal system with impulse response $h[n]$, we can define the partial energy of the impulse response as

$$\mathcal{E}[n] = \sum_{k=0}^{n} |h[k]|^2.$$

A minimum phase system with impulse response $h_{\min}[n]$ has the property

$$\mathcal{E}_{\min}[n] = \sum_{k=0}^{n} |h_{\min}[k]|^2 \geq \sum_{k=0}^{n} |h[k]|^2 = \mathcal{E}[n]$$

for all $n \geq 0$ and all $h[n]$ with the same magnitude response as $h_{\min}[n]$.

Recall our earlier example:

$$\begin{aligned}
H_1(z) &= 6 + z^{-1} - z^{-2} & \mathcal{E} &= \{36, 37, 38\} \\
H_2(z) &= 1 - z^{-1} - 6z^{-2} & \mathcal{E} &= \{1, 2, 38\} \\
H_3(z) &= 2 - 5z^{-1} - 3z^{-2} & \mathcal{E} &= \{4, 29, 38\} \\
H_4(z) &= 3 + 5z^{-1} - 2z^{-2} & \mathcal{E} &= \{9, 34, 38\}
\end{aligned}$$