Digital Signal Processing
Minimum-Phase All-Pass Decomposition

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Minimum-Phase All-Pass Decomposition

Suppose we have a causal stable rational transfer function $H(z)$ with one or more zeros outside the unit circle. We denote the zeros outside the unit circle as $\{c_1, \dots, c_M\}$.

We can form a minimum phase system with the same magnitude response by reflecting these poles to their conjugate symmetric locations inside the unit circle, i.e.,

$$H_{\min}(z) = H(z) \cdot \underbrace{\frac{z^{-1} - c_1^*}{1 - c_1 z^{-1}} \cdots \frac{z^{-1} - c_M^*}{1 - c_M z^{-1}}}_{\text{unit-magnitude all-pass filter}}$$

It should be clear that $H_{\min}(z)$ and $H(z)$ have the same magnitude response. Moreover, we have the decomposition

$$H(z) = H_{\min}(z) \cdot \left(\frac{z^{-1} - c_1^*}{1 - c_1 z^{-1}} \cdots \frac{z^{-1} - c_M^*}{1 - c_M z^{-1}} \right)^{-1} = H_{\min}(z) H_{\text{ap}}(z)$$

where $H_{\text{ap}}(z)$ is an all-pass filter.

Example 1 (part 1 of 2)

Suppose

$$H(z) = \frac{1 - 2z^{-1}}{1 + \frac{1}{3}z^{-1}}$$

This system is clearly not minimum phase since it has a zero at $z = 2$. We can reflect this zero inside the unit circle with an all-pass filter to write

$$\begin{aligned} H_{\min}(z) &= H(z) \cdot \frac{z^{-1} - 2}{1 - 2z^{-1}} \\ &= \frac{z^{-1} - 2}{1 + \frac{1}{3}z^{-1}} \end{aligned}$$

hence

$$H(z) = \underbrace{\frac{z^{-1} - 2}{1 + \frac{1}{3}z^{-1}}}_{H_{\min}(z)} \cdot \underbrace{\frac{1 - 2z^{-1}}{z^{-1} - 2}}_{H_{\text{ap}}(z)}$$

Example 1 (part 2 of 2)

Note that, just requiring the zeros to be inside the unit circle does not uniquely specify $H_{\min}(z)$. For example, we could also write

$$H(z) = \underbrace{\frac{2 - z^{-1}}{1 + \frac{1}{3}z^{-1}}}_{G_{\min}(z)} \cdot \underbrace{\frac{2z^{-1} - 1}{z^{-1} - 2}}_{G_{\text{ap}}(z)}$$

where $G = -H$ for both systems. Does it matter which one we pick?

To satisfy the minimum phase delay property, recall that we require $\angle H_{\min}(e^{j0}) = 0$. Note that

$$H_{\min}(e^{j0}) = \frac{1 - 2}{1 + \frac{1}{3}} < 0 \text{ and } G_{\min}(e^{j0}) = \frac{2 - 1}{1 + \frac{1}{3}} > 0$$

hence $\angle H_{\min}(e^{j0}) = -\pi$ and $\angle G_{\min}(e^{j0}) = 0$. The correct answer is to choose the $G_{\min}(z)G_{\text{ap}}(z)$ decomposition.

Example 2

Suppose

$$H(z) = \frac{(1+3z^{-1})(1-\frac{1}{2}z^{-1})}{z^{-1}(1+\frac{1}{3}z^{-1})}$$

This is also clearly not minimum phase due to the zero at $z=-3$. We can reflect this zero inside the unit circle to write

$$\begin{aligned} H_{\min}(z) &= H(z) \cdot \frac{z^{-1}+3}{1+3z^{-1}} \\ &= \frac{(z^{-1}+3)(1-\frac{1}{2}z^{-1})}{z^{-1}(1+\frac{1}{3}z^{-1})} \\ &= \frac{3(1-\frac{1}{2}z^{-1})}{z^{-1}} \end{aligned}$$

Recall that minimum phase filters must be causal. Since $H_{\min}(z) = 3z - \frac{1}{2}$ is clearly not causal, we can factor out the z^{-1} denominator term (putting it in the all-pass filter since it does not affect the magnitude response) to arrive at

$$H(z) = \underbrace{3\left(1-\frac{1}{2}z^{-1}\right)}_{H_{\min}(z)} \cdot \underbrace{\frac{z^{-1}+3}{z^{-1}(1+3z^{-1})}}_{H_{\text{ap}}(z)}$$

Equalization of Nonminimum Phase Channel

Suppose $H_1(z) = \frac{(z-4)(z+5)}{(z+0.5)(z-0.3)}$ with ROC $|z| > 0.5$.

We form the inverse system $H_2(z) = \frac{(z+0.5)(z-0.3)}{(z-4)(z+5)}$. Note that there is no causal stable inverse here.

One approach in this case is to factor $H_1(z)$ into a causal stable minimum phase filter and a causal stable allpass filter, i.e.

$$H_1(z) = H_{\min}(z)H_{\text{ap}}(z) = \underbrace{\frac{(4z-1)(5z+1)}{(z+0.5)(z-0.3)}}_{\text{minimum phase}} \underbrace{\frac{(z-4)(z+5)}{(4z-1)(5z+1)}}_{\text{allpass}} \quad \text{ROC} : |z| > 0.5$$

and invert just the minimum phase component, i.e. $H_2(z) = \frac{(z+0.5)(z-0.3)}{(4z-1)(5z+1)}$.

Then $H_1(z)H_2(z) \neq 1$ but rather $H_1(z)H_2(z) = H_{\text{ap}}(z)$. Hence, the equalizer $H_2(z)$ corrects the magnitude distortion, but leaves some residual phase distortion.