Digital Signal Processing
FIR Filters with Generalized Linear Phase

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Generalized Linear Phase

Definition

A system has **linear phase** if its phase response $\theta(\omega) = \angle H(e^{j\omega}) = -c\omega$ for all ω and any constant c.

In general, a linear phase system has frequency response

$$H(e^{j\omega}) = |H(e^{j\omega})|e^{-j\omega c}$$

and delays all frequencies by the same amount of time.

Definition

A system has **generalized linear phase** if its frequency response can be written as $H(e^{j\omega}) = A(e^{j\omega})e^{-j\omega c + jb}$ where c and b are constants and $A(e^{j\omega})$ is a real (not necessarily positive) function.

GLP systems are sometimes called "affine phase" systems and have constant group delay except at discontinuities in the phase response.

Type I Causal FIR Generalized Linear-Phase Systems

Characteristics:

- Odd number of coefficients.
- Symmetric

Example (filter order $M = 2$): $h[n] = \{\underline{1}, 2, 1\}$.

Frequency response

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= h[0](e^{-j\omega 0} + e^{-j\omega M}) + h[1](e^{-j\omega 1} + e^{-j\omega(M-1)}) + \cdots + h[M/2]e^{-j\omega M/2} \\
&= e^{-j\omega M/2}\left(h[0](e^{j\omega M/2} + e^{-j\omega M/2}) + \cdots + h[M/2]\right) \\
&= e^{-j\omega M/2}\left(h[0] \cdot 2\cos(\omega M/2) + h[1] \cdot 2\cos(\omega(M/2-1)) \cdots + h[M/2]\right) \\
&= e^{-j\omega M/2} \sum_{k=0}^{M/2} a_1[k]\cos(\omega k)
\end{aligned}
$$

where

$$
a_1[k] = \begin{cases} h[M/2] & k = 0 \\ 2h[M/2-k] & \text{otherwise.} \end{cases}
$$

Type II Causal FIR Generalized Linear-Phase Systems

Characteristics:

- Even number of coefficients.
- Symmetric

Example (filter order $M = 3$): $h[n] = \{\underline{1}, 2, 2, 1\}$.

Frequency response

$$
\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n] e^{-j\omega n} \\
&= h[0](e^{-j\omega 0} + e^{-j\omega M}) + \cdots + h[(M-1)/2](e^{-j\omega(M-1)/2} + e^{-j\omega(M+1)/2}) \\
&= e^{-j\omega M/2}\left(h[0](e^{j\omega M/2} + e^{-j\omega M/2}) + \cdots + h[(M-1)/2](e^{j\omega/2} + e^{-j\omega/2})\right) \\
&= e^{-j\omega M/2}\left(h[0] \cdot 2\cos(\omega M/2) + \cdots + h[(M-1)/2] \cdot 2\cos(\omega/2)\right) \\
&= e^{-j\omega M/2} \sum_{k=0}^{(M-1)/2} a_2[k] \cos\left(\omega\left(k + \frac{1}{2}\right)\right)
\end{aligned}
$$

where

$$
a_2[k] = 2h[(M-1)/2 - k].
$$

Type III Causal FIR Generalized Linear-Phase Systems

Characteristics:

- Odd number of coefficients.
- Anti-symmetric

Example (filter order $M = 2$): $h[n] = \{\underline{1}, 0, -1\}$ ($h[M/2]$ must be zero).

Frequency response

$$\begin{aligned} H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\ &= h[0](e^{-j\omega 0} - e^{-j\omega M}) + \dots + h[1](e^{-j\omega 1} - e^{-j\omega(M-1)}) + \dots + h[M/2]e^{-j\omega M/2} \\ &= e^{-j\omega M/2}\left(h[0](e^{j\omega M/2} - e^{-j\omega M/2}) + h[1](e^{j\omega(M/2-1)} - e^{-j\omega(M/2-1)}) + \dots\right) \\ &= e^{-j\omega M/2}\left(h[0] \cdot 2j\sin(\omega M/2) + h[1] \cdot 2j\sin(\omega(M/2-1)) + \dots\right) \\ &= je^{-j\omega M/2} \sum_{k=0}^{M/2-1} a_3[k]\sin(\omega(k+1)) \end{aligned}$$

where

$$a_3[k] = 2h[M/2 - k - 1].$$

Type IV Causal FIR Generalized Linear-Phase Systems

Characteristics:

- Even number of coefficients.
- Anti-symmetric

Example (filter order $M = 3$): $h[n] = \{\underline{1}, -2, 2, -1\}$.

Frequency response

$$\begin{aligned}
H(e^{j\omega}) &= \sum_{n=0}^{M} h[n]e^{-j\omega n} \\
&= h[0](e^{-j\omega 0} - e^{-j\omega M}) + \cdots + h[(M-1)/2](e^{-j\omega(M-1)/2} - e^{-j\omega(M+1)/2}) \\
&= e^{-j\omega M/2}\left(h[0](e^{j\omega M/2} - e^{-j\omega M/2}) + \cdots + h[(M-1)/2](e^{j\omega/2} - e^{-j\omega/2})\right) \\
&= e^{-j\omega M/2}\left(h[0] \cdot 2j\sin(\omega M/2) + \cdots + h[(M-1)/2] \cdot 2j\sin(\omega/2)\right) \\
&= je^{-j\omega M/2} \sum_{k=0}^{(M-1)/2} a_4[k] \sin\left(\omega\left(k + \frac{1}{2}\right)\right)
\end{aligned}$$

where

$$a_4[k] = 2h[(M-1)/2 - k].$$

Summary

Type	Impulse response symmetry	Impulse response length
I	symmetric	$M+1$ odd (order is even)
II	symmetric	$M+1$ even (order is odd)
III	antisymmetric	$M+1$ odd (order is even)
IV	antisymmetric	$M+1$ even (order is odd)

Type	LPF	HPF	BPF	BSF	Comment
I	Y	Y	Y	Y	Most versatile.
II	Y	N	Y	N	Zero at $z=-1$.
III	N	N	Y	N	Zeros at $z=\pm 1$.
IV	N	Y	Y	N	Zero at $z=1$.

All have the same constant group delay, $\tau_g(\omega)=\frac{M}{2}$, which is an integer for Type I and Type III but is not an integer for Type II and Type IV.