Digital Signal Processing
FIR Cascaded Lattice Filters

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FIR Cascaded Lattice Structures

This approach can realize two FIR transfer functions:

\[ H(z) = \frac{Y(z)}{X(z)} \quad \text{and} \quad G(z) = \frac{W(z)}{X(z)}. \]

Given a lattice structure block diagram, how do we determine \( H(z) \) and \( G(z) \) (lattice \( \rightarrow \) TF)?

Given a desired FIR \( H(z) \) and \( G(z) \) (or \( h[n] \) and \( g[n] \)), how do we determine the lattice coefficients (TF \( \rightarrow \) lattice)?
FIR Cascaded Lattice Structures (Lattice → TF)

Analysis of single lattice segment:

\[ H_\ell(z) = H_{\ell-1}(z) + z^{-1}\delta_\ell G_{\ell-1}(z) \]
\[ G_\ell(z) = \gamma_\ell H_{\ell-1}(z) + z^{-1}G_{\ell-1}(z) \]

Since \( H_0(z) = \delta_0 X(z) \) and \( G_0(z) = \gamma_0 X(z) \), this recursion can be applied for \( \ell = 1, 2, \ldots, N \) to determine the transfer functions \( H(z) \) and \( G(z) \) from the lattice coefficients.
We can write

\[ H_0(z) = \delta_0 \]
\[ G_0(z) = \gamma_0 \]
\[ H_1(z) = H_0(z) + \delta_1 z^{-1} G_0(z) = \delta_0 + \delta_1 z^{-1} \gamma_0 \]
\[ G_1(z) = \gamma_1 H_0(z) + z^{-1} G_0(z) = \gamma_1 \delta_0 + z^{-1} \gamma_0 \]
\[ H_2(z) = H_1(z) + \delta_2 z^{-1} G_1(z) = \delta_0 + \delta_1 z^{-1} \gamma_0 + \delta_2 z^{-1} (\gamma_1 \delta_0 + z^{-1} \gamma_0) \]
\[ G_2(z) = \gamma_2 H_1(z) + z^{-1} G_1(z) = \gamma_2 (\delta_0 + \delta_1 z^{-1} \gamma_0) + z^{-1} (\gamma_1 \delta_0 + z^{-1} \gamma_0) \]
and so on.
To get a recursion for computing lattice coefficients given $H(z)$ and $G(z)$, we can redo our analysis, solving instead for $H_{\ell-1}(z)$ and $G_{\ell-1}(z)$, to get

$$H_{\ell-1}(z) = K_\ell \left[ H_\ell(z) - \delta_\ell G_\ell(z) \right]$$

$$G_{\ell-1}(z) = K_\ell z \left[ G_\ell(z) - \gamma_\ell H_\ell(z) \right]$$

where $K_\ell = \frac{1}{1 - \delta_\ell \gamma_\ell}$. 
FIR Cascaded Lattice Structures (TF → Lattice)

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where $K_\ell = \frac{1}{1-\delta_\ell \gamma_\ell}$.

Suppose, as an example, $\ell = 2$ with $H_\ell(z) = h_0 + h_1 z^{-1} + h_2 z^{-2}$ and $G_\ell(z) = g_0 + g_1 z^{-1} + g_2 z^{-2}$. Then

$$H_1(z) = K_2 [h_0 + h_1 z^{-1} + h_2 z^{-2} - \delta_2 (g_0 + g_1 z^{-1} + g_2 z^{-2})]$$

$$G_1(z) = K_2 z [g_0 + g_1 z^{-1} + g_2 z^{-2} - \gamma_2 (h_0 + h_1 z^{-1} + h_2 z^{-2})]$$

Note that $H_1(z)$ and $G_1(z)$ can only be order 1 transfer functions. Hence we need the $z^{-2}$ and $z^{+1}$ terms to vanish, which happens when

$z^{-2}$ term in $H_1(z)$ vanishes $\iff h_2 - \delta_2 g_2 = 0 \iff \delta_2 = \frac{h_2}{g_2}$

$z^{+1}$ term in $G_1(z)$ vanishes $\iff g_0 - \gamma_2 h_0 = 0 \iff \gamma_2 = \frac{g_0}{h_0}$
**FIR Cascaded Lattice Structures (TF → Lattice)**

Outline of procedure:

1. Set $L$ equal to the filter order (impulse response length minus 1).
2. Set $H_L(z) = H(z)$ and $G_L(z) = G(z)$.
3. Set $\ell = L$.
4. Solve for $\delta_\ell$ and $\gamma_\ell$.
5. Compute $K_\ell = \frac{1}{1-\delta_\ell \gamma_\ell}$.
6. Compute $H_{\ell-1}(z)$ and $G_{\ell-1}(z)$. Both transfer functions should be order $\ell - 1$. Note coefficients are affected by $K_\ell$ (you can’t just say $H_{\ell-1}(z)$ is equal to the first $\ell - 1$ coefficients of $H_\ell(z)$ or that $G_{\ell-1}(z)$ is equal to the first $\ell - 1$ coefficients of $G_\ell(z)$).
7. Decrement $\ell$.
8. If $\ell \geq 1$ go to step 4.
9. The final gains at the input are simply $\delta_0 = H_0(z)$ and $\gamma_0 = G_0(z)$ (these are both zero-order transfer functions).
Suppose $H(z) = 1 + 2z^{-1} + 3z^{-2}$ and $G(z) = 4 + 5z^{-1} + 6z^{-2}$.

1. We set $\ell = L = 2$ and solve $\delta_2 = \frac{3}{6} = \frac{1}{2}$ and $\gamma_2 = \frac{4}{1} = 4$.

2. We compute $K_2 = \frac{1}{1-2} = -1$.

3. We compute

$$H_1(z) = -1 \cdot \left[ 1 + 2z^{-1} + 3z^{-2} - \frac{1}{2}(4 + 5z^{-1} + 6z^{-2}) \right] = 1 + \frac{1}{2}z^{-1}$$

and

$$G_1(z) = -1z \left[ 4 + 5z^{-1} + 6z^{-2} - 4(1 + 2z^{-1} + 3z^{-2}) \right] = 3 + 6z^{-1}$$

4. Now set $\ell = 1$ and solve $\delta_1 = \frac{\frac{1}{2}}{6} = \frac{1}{12}$ and $\gamma_1 = \frac{3}{1} = 3$.

5. Compute $K_1 = \frac{\frac{1}{1}}{1-\frac{3}{12}} = \frac{4}{3}$.

6. Compute

$$H_0(z) = \frac{4}{3} \cdot \left[ 1 + \frac{1}{2}z^{-1} - \frac{1}{12}(3 + 6z^{-1}) \right] = 1 = \delta_0$$

and

$$G_0(z) = \frac{4}{3}z \left[ 3 + 6z^{-1} - 3(1 + \frac{1}{2}z^{-1}) \right] = 6 = \gamma_0$$
Often, we only need a single transfer function and can use the symmetric lattice

The procedures for TF → lattice and lattice → TF are similar to what we’ve covered. Also see Matlab functions tf2latc and latc2tf.