# Digital Signal Processing
# Algebraic Method for
# Computation of Output Noise Variance

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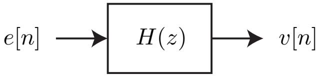
# Propagation of White Noise to Filter Output

Consider the situation with $e[n]$ a white noise sequence, e.g. $e[n]$ represents quantization noise:




We assume $H(z)$ is a causal stable real rational transfer function. Since $H(z)$ is an LTI system and $\{e[n]\}$ is a zero-mean independent random sequence, i.e., white noise, we know:

- Output mean:
$$m_v = H(e^{j0})m_e = 0$$
- Output variance:
$$\sigma_v^2 = \sigma_e^2 \cdot \frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 \, d\omega = \sigma_e^2 \cdot \sum_{n=-\infty}^{\infty} |h[n]|^2$$

It can sometimes be difficult to compute the integral or the infinite sum.

# Equivalent Expression for Output Noise Variance (1 of 2)

It can be shown [Mitra, Digital Signal Processing, 4th Edition] that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 \, d\omega = \frac{1}{2\pi j}\oint_C H(z)H(z^{-1})z^{-1}\, dz$$

where $C$ is a counterclockwise contour in the ROC of $H(z)H(z^{-1})$.

Since $H(z)$ is a causal stable real rational transfer function, it can be expressed as a partial-fraction sum

$$H(z) = \sum_{i=1}^{R} H_i(z)$$

where each $H_i(z)$ is a low-order (typically zeroth-order or first-order) transfer function.

# Equivalent Expression for Output Noise Variance (2 of 2)

We can substitute this last result to write

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|H(e^{j\omega})|^2\,d\omega = \frac{1}{2\pi j}\sum_{k=1}^{R}\sum_{\ell=1}^{R}\oint_C H_k(z)H_\ell(z^{-1})z^{-1}\,dz$$

It can be further shown that

$$\oint_C H_k(z)H_\ell(z^{-1})z^{-1}\,dz = \oint_C H_\ell(z)H_k(z^{-1})z^{-1}\,dz$$

hence

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi}|H(e^{j\omega})|^2\,d\omega &= \frac{1}{2\pi j}\Big\{\sum_{k=1}^{R}\oint_C H_k(z)H_k(z^{-1})z^{-1}\,dz \\ &\qquad + 2\sum_{k=1}^{R}\sum_{\ell=k+1}^{R}\oint_C H_k(z)H_\ell(z^{-1})z^{-1}\,dz\Big\} \\ &= \sum_{k=1}^{R} J_{k,k} + 2\sum_{k=1}^{R}\sum_{\ell=k+1}^{R} J_{k,\ell}\end{aligned}$$

# Constituent Terms

We have

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}|H(e^{j\omega})|^2\,d\omega = \sum_{k=1}^{S} J_{k,k} + 2\sum_{k=1}^{R}\sum_{\ell=k+1}^{R} J_{k,\ell}$$

with

$$J_{k,\ell} = \frac{1}{2\pi j}\oint_C H_k(z)H_\ell(z^{-1})z^{-1}\,dz$$

After a partial fraction expansion, typical $H_k(z)$ will be

$$A \qquad \text{or} \qquad \frac{B_k}{z-a_k} \qquad \text{or} \qquad \frac{B_k z + D_k}{z^2 + b_k z + d_k}$$

If all of the $H_k(z)$ match these forms, it turns out that these "small" contour integrals are not too difficult to compute with Cauchy's residue theorem. In fact, the results have been tabulated [Mitra] ...

From Table 12.4 in [Mitra, Digital Signal Processing, 4th edition]:

| $H_k(z)$ | $H_\ell(z^{-1})$: $A$ | $H_\ell(z^{-1})$: $\frac{B_\ell}{z^{-1}-a_\ell}$ | $H_\ell(z^{-1})$: $\frac{C_\ell z^{-1}+D_\ell}{z^{-2}+b_\ell z^{-1}+d_\ell}$ |
|---|---|---|---|
| $A$ | $I_1$ | $0$ | $0$ |
| $\frac{B_k}{z-a_k}$ | $0$ | $I_2$ | $I_4'$ |
| $\frac{C_k z+D_k}{z^2+b_k z+d_k}$ | $0$ | $I_4$ | $I_3$ |

From Table 12.4 in [Mitra, Digital Signal Processing, 4th edition]:

$$I_1 = A^2$$

$$I_2 = \frac{B_k B_\ell}{1 - a_k a_\ell}$$

$$I_3 = \frac{(C_k C_\ell + D_k D_\ell)(1 - d_k d_\ell) - (C_\ell D_k - D_\ell C_k d_k) b_\ell - (C_k D_\ell - D_k C_\ell d_\ell) b_k}{(1 - d_k d_\ell)^2 + d_k b_\ell^2 + d_\ell b_k^2 - (1 + d_k d_\ell) b_k b_\ell}$$

$$I_4 = \frac{B_\ell (C_k + D_k a_\ell)}{1 + b_k a_\ell + d_k a_\ell^2}$$

$$I_4' = \frac{B_k (C_\ell + D_\ell a_k)}{1 + b_\ell a_k + d_\ell a_k^2}$$

# Simple Example

Suppose $H(z) = \frac{1}{1-\alpha z^{-1}}$ with $|\alpha| < 1$ and ROC $z > |\alpha|$. The easiest way to compute $\frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 \, d\omega$ is probably to take the inverse $z$-transform and use the series, but let's illustrate the algebraic method here.

We can write

$$H(z) = \frac{z}{z-\alpha} = 1 + \frac{\alpha}{z-\alpha}$$

which then has terms in the required form for the algebraic method: $H_1(z) = 1$ and $H_2(z) = \frac{\alpha}{z-\alpha}$. We can now use the table to say

$$\begin{aligned}\frac{1}{2\pi}\int_{-\pi}^{\pi} |H(e^{j\omega})|^2 \, d\omega &= J_{1,1} + J_{2,2} + 2J_{2,1} \\ &= I_1 + I_2 + 0 \\ &= 1^2 + \frac{\alpha^2}{1-\alpha^2} \\ &= \frac{1}{1-\alpha^2}.\end{aligned}$$