ECE503 Spring 2014 Quiz 2

Your Name: _____________________________  ECE Box Number: _______

Instructions: This quiz is worth a total of 100 points. The quiz is open book and open notes. You may also use a calculator. You may not use a computer, phone, or tablet. Please show your work on each problem and box/circle your final answers. Points may be deducted for a disorderly presentation of your solution.

1. 30 points. Compute the $z$-transform and the region of convergence (ROC) of the sequence defined as

$$x[n] = \begin{cases} 
1 & n = 0, 2, 4, \ldots \\
0 & \text{otherwise.}
\end{cases}$$

Simplify your answer as much as possible to receive full credit.

2. 50 points total. Given

$$X(z) = \frac{3 - z^{-1}}{1 - 3z^{-1} + 2z^{-2}}.$$

(a) 10 points. List all of the poles and zeros of $X(z)$.
(b) 20 points. Determine all possible regions of convergence for $X(z)$.
(c) 20 points. Compute the sequence $x[n]$ corresponding to each region of convergence.

3. 20 points. Recall that the $z$-transform of the unit-step function $x[n] = u[n]$ is given as

$$X(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1$$

and the DTFT of the unit step function is given as

$$X(e^{j\omega}) = \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) + \frac{1}{1 - e^{-j\omega}}.$$

Note that $X(e^{j\omega}) \neq X(z)|_{z=e^{j\omega}}$ in this case. Explain why.
1. \[ X(z) = \sum_{n=0}^{\infty} x[n]z^{-n} = \sum_{n=0}^{\infty} z^{-n} = \sum_{m=0}^{\infty} z^{-2m} = \sum_{m=0}^{\infty} (z^2)^{-m} \]

\[ (1-z^{-2}) \sum_{m=0}^{N} (z^2)^{-m} = (1-z^{-2}) \left[ 1 + z^{-2} + z^{-4} + \ldots + z^{-2N} \right] \]

\[ = 1 - z^{-2N-2} \]

so \[ \sum_{m=0}^{N} (z^2)^{-m} = \frac{1 - z^{-2N-2}}{1-z^{-2}} \]

and \[ \sum_{m=0}^{\infty} (z^2)^{-m} = \frac{1}{1-z^{-2}} \quad \text{if } |z| > 1 \]

Hence \[ X(z) = \frac{1}{1-z^{-2}} \quad \text{with ROC } |z| > 1 \]

2. a) \[ X(z) = \frac{3-z^{-1}}{(1-2z^{-1})(1-z^{-1})} \]

poles at \( z = 1 \) and \( z = 2 \)

zeros at \( z = \frac{1}{3} \) and \( z = 0 \)

To see the pole at \( z = 0 \), you can rewrite \( X(z) \) as \[ X(z) = \frac{3z^2-z}{(z-2)(z-1)} \]

Clearly \( X(z) = 0 \) when \( z = 0 \)

b) \[
\begin{align*}
\text{ROC}_A & : |z| < 1 \\
\text{ROC}_B & : 1 < |z| < 2 \\
\text{ROC}_C & : |z| > 2 \\
\end{align*}
\]

Each ROC is bounded by the poles

\( \text{ROC}_A \rightarrow \text{left-sided sequence} \)

\( \text{ROC}_B \rightarrow \text{two-sided sequence} \)

\( \text{ROC}_C \rightarrow \text{right-sided sequence} \)
c) partial fraction expansion

\[ X(z) = \frac{A_1}{1-2z^{-1}} + \frac{A_2}{1-z^{-1}} \]

\[ A_1 = (1-2z^{-1})X(z)|_{z=2} = \frac{3-2^{-1}}{1-2^{-1}} \bigg|_{z=2} = \frac{3-\frac{1}{2}}{1-\frac{1}{2}} = 5 \]

\[ A_2 = (1-z^{-1})X(z)|_{z=1} = \frac{3-2^{-1}}{1-2^{-1}} \bigg|_{z=1} = \frac{3-1}{1-2} = -2 \]

- \[ \text{ROC}_A : |z| < 1 \Rightarrow x[n] = -5 \cdot 2^n u[-n-1] + 2 \cdot u[-n+1] \]
- \[ \text{ROC}_B : |1| < |z| < 2 \Rightarrow x[n] = -5 \cdot 2^n u[-n-1] - 2 u[n] \]
- \[ \text{ROC}_C : |z| > 2 \Rightarrow x[n] = 5 \cdot 2^n u[n] - 2 u[n] \]

3. \[ X(e^{j\omega}) = X(z)|_{z=e^{j\omega}} \] only when the ROC of \( X(z) \) includes the unit circle. That is not the case here, since the ROC is \( |z| > 1 \).

In general, \( X(z) \) is an analytic function of \( z \) and hence can't have discontinuous functions like \( \delta(z) \).