ECE503 Spring 2014 Quiz 5

Your Name: ______________________________ ECE Box Number: ________

Instructions: This quiz is worth a total of 100 points. The quiz is open book and open notes. You may also use a calculator. You may not use a computer, phone, or tablet. Please show your work on each problem and box/circle your final answers. Points may be deducted for a disorderly presentation of your solution.

1. 50 points total. Suppose you have a continuous-time signal $x_c(t)$ applied to the input of the system shown in Fig. 1 with ideal continuous/discrete and discrete/continuous blocks. Note that the sampling period of the C/D block is T whereas the sampling period of the D/C block is MT for integer $M \geq 1$. Further suppose the anti-aliasing filter $H_a(j\Omega)$ is a zero-phase filter with magnitude response shown in Fig. 2 with $\Omega_1 = 2\pi \cdot 5000$ and $\Omega_2 = 2\pi \cdot 19000$ (the figure is not to scale).

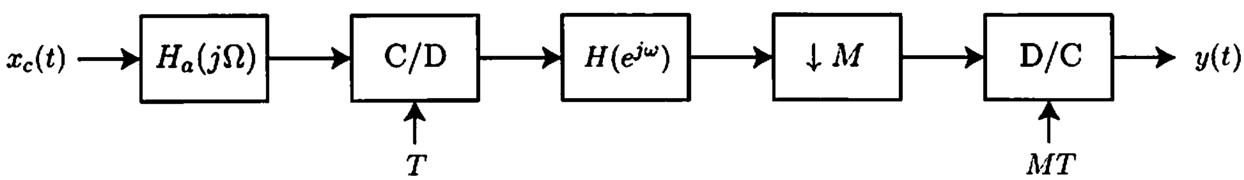


Figure 1: System for processing $x_c(t)$.

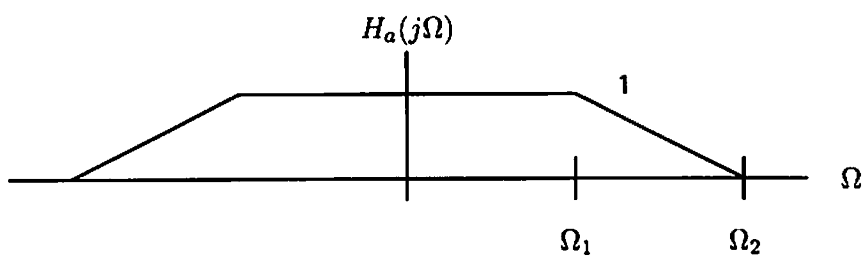


Figure 2: Anti-aliasing filter spectrum.

We desire the overall system from $x_c(t)$ to $y(t)$ to be an ideal lowpass filter with cutoff frequency $\Omega_c = 2\pi \cdot 4000$.

(a) 20 points. What is the minimum sampling frequency $f_s = \frac{1}{T}$ that can be used to achieve the desired overall response for any input? Explain.

(b) 30 points. Suppose $f_s = 100$ kHz. Specify discrete-time system $H(e^{j\omega})$ that achieves the desired overall response. What is the maximum value of M that can be used without affecting the desired overall response? Explain.

2. 50 points. Consider an oversampled ADC with noise shaping, modeled as shown in Fig. 3. The oversampled discrete-time signal $x[n]$ is assumed to be zero mean and stationary with variance σ_x^2. The discrete-time quantization noise $e[n]$ is assumed to be zero mean, white, stationary with variance σ_e^2, and uncorrelated with $x[n]$. The input to the system is also assumed to be band limited to Ω_N so that the output of the overall system can be written as

$$x_d[n] = x[n] + f[n]$$

where $f[n]$ is the quantization noise at the output of the system. Suppose the system that shapes the noise ($H(z)$) has magnitude response as shown in Fig. 4. Determine the signal to quantization noise ratio (SQNR) at the output of this system as a function of σ_e^2, σ_x^2, and M. Compare your result to the SQNR of conventional noise shaping with $H(z) = 1 - z^{-1}$.

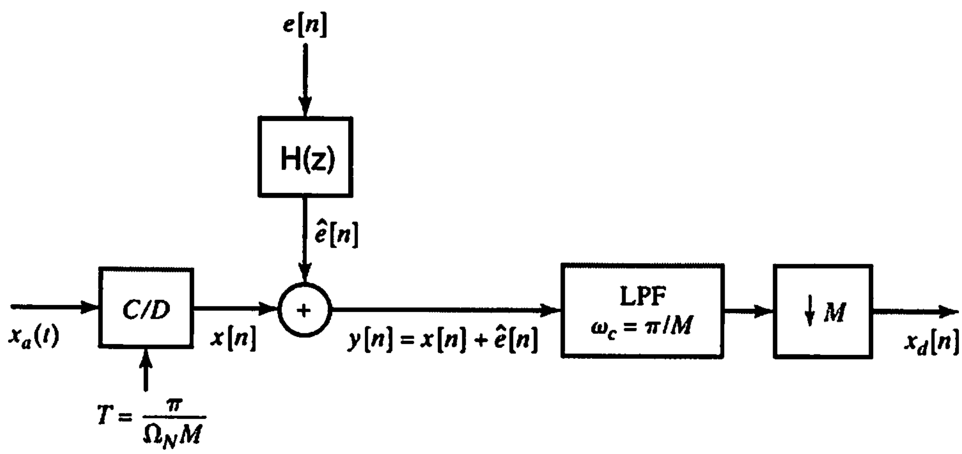


Figure 3: Oversampled ADC with noise shaping (from O & S textbook).

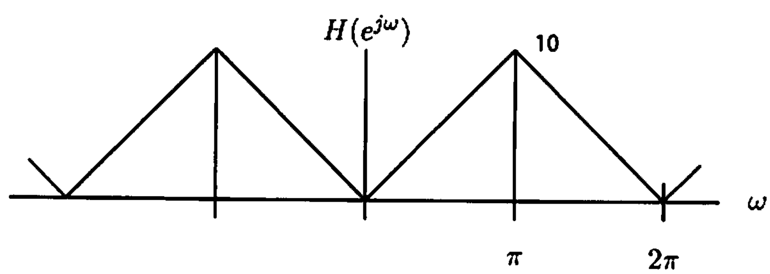
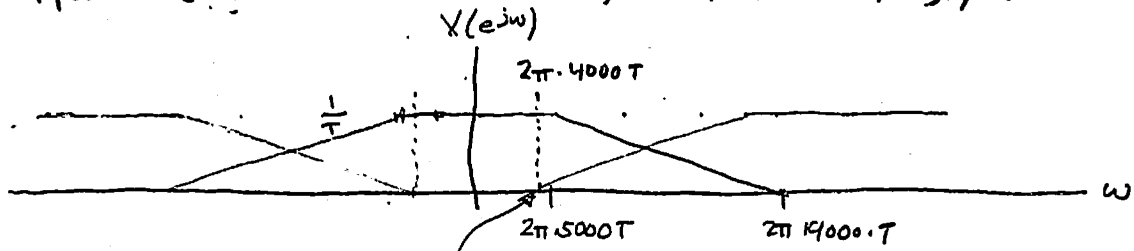


Figure 4: Magnitude response of noise shaping filter.

1.a) Suppose $x_c(t)$ is a white noise signal. After sampling, we have



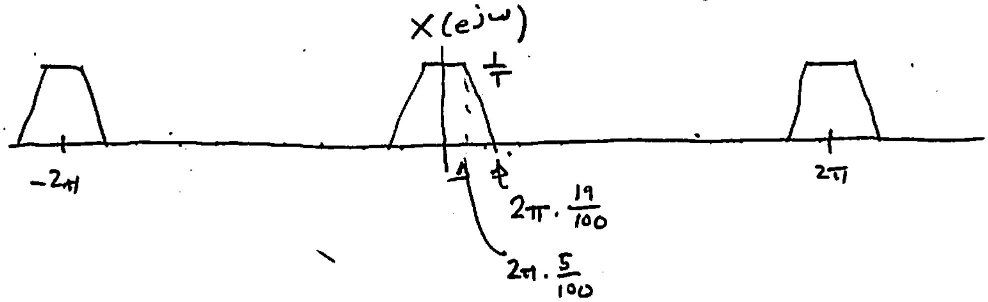
this point is $2\pi - 2\pi \cdot 19000T$

We need $2\pi - 2\pi \cdot 19000T \geq 2\pi \cdot 4000T$ to avoid aliasing into the passband of our overall filter.

$$1 \geq 23000T$$

so $\boxed{f_s \geq 23000}$ or $T \leq \frac{1}{23000}$

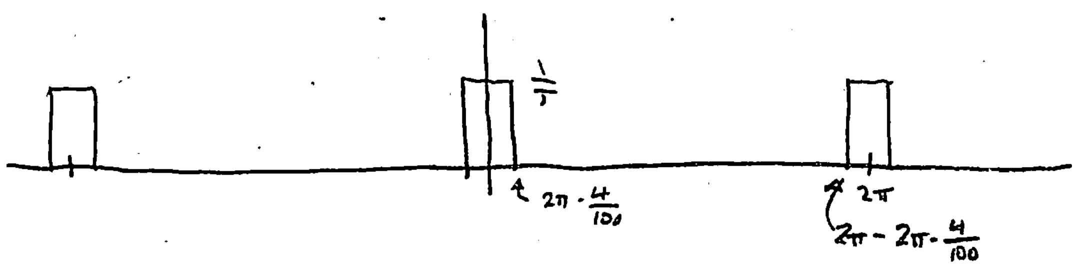
b) If $f_s = 100\text{ kHz}$ then after sampling we have



We can apply an ideal DT lowpass filter here with cutoff

$$\omega_c = \frac{2\pi \cdot 4}{100}$$

and the output of this filter will be



When we down sample, to avoid aliasing, we need

$$2\pi \cdot \frac{4}{100} \cdot M \leq 2\pi - 2\pi \cdot \frac{4M}{100}$$

$$\frac{M}{25} \leq 1 - \frac{M}{25} \Rightarrow \frac{2M}{25} \leq 1 \qquad \boxed{M \leq 12}$$

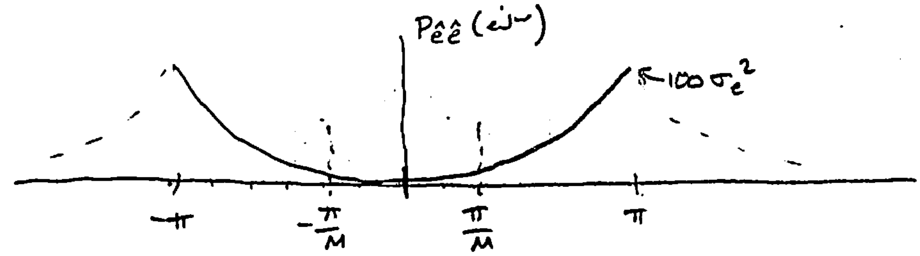
2. First look at the PSD of $\hat{e}[n]$

$$P_{\hat{e}\hat{e}}(e^{j\omega}) = |H(e^{j\omega})|^2 \, P_{ee}(e^{j\omega})$$

For $\pi \le \omega \le \pi$, we have $|H(e^{j\omega})| = \frac{10 \cdot |\omega|}{\pi}$

so

$$P_{\hat{e}\hat{e}}(e^{j\omega}) = \sigma_e^2 \cdot \frac{100\,\omega^2}{\pi^2}$$



Now, when we pass this through the LPF, the quantization noise power at the output of the LPF is

$$\sigma_v^2 = \frac{1}{2\pi}\int_{-\pi/M}^{\pi/M} \sigma_e^2 \,\frac{100\,\omega^2}{\pi^2}\, d\omega = \frac{1}{2\pi}\cdot\sigma_e^2\cdot\frac{100}{\pi^2}\int_{-\pi/M}^{\pi/M}\omega^2\, d\omega$$

$$= \frac{50\,\sigma_e^2}{\pi^3}\cdot 2\,\frac{\omega^3}{3}\Big|_0^{\pi/M} = \frac{50\,\sigma_e^2}{\cancel{\pi^3}}\cdot\frac{2}{3}\,\frac{\cancel{\pi^3}}{M^3}$$

$$= \frac{100\,\sigma_e^2}{3M^3} \approx 33.3\,\frac{\sigma_e^2}{M^3}$$

The downsampler doesn't change the power, so $\sigma_f^2 = \sigma_v^2$ and we have

$$SQNR = 10\log_{10}\left(\frac{\sigma_x^2}{\sigma_f^2}\right) = 10\log_{10}\left(\frac{\sigma_x^2\cdot 3M^3}{100\cdot\sigma_e^2}\right)$$

$$= 10\log_{10}\left(\frac{3\,\sigma_x^2}{100\,\sigma_e^2}\right) + 30\log_{10}(M)$$

As was the case with the usual noise shaping $H(z) = 1 - z^{-1}$, we have a $30\log_{10}(M)$ term. The quantization noise variance in that case, however, was

$$\sigma_v^2 = \frac{2\pi^2}{6M^3}\cdot\sigma_e^2 \approx 3.3\,\frac{\sigma_e^2}{M^3}$$

- which is ~10 times better than the system considered here.