ECE503 Spring 2014 Quiz 5

Your Name: ________________________ ECE Box Number: ________

Instructions: This quiz is worth a total of 100 points. The quiz is open book and open notes. You may also use a calculator. You may not use a computer, phone, or tablet. Please show your work on each problem and box/circle your final answers. Points may be deducted for a disorderly presentation of your solution.

1. 50 points total. Suppose you have a continuous-time signal \( x_c(t) \) applied to the input of the system shown in Fig. 1 with ideal continuous/discrete and discrete/continuous blocks. Note that the sampling period of the C/D block is \( T \) whereas the sampling period of the D/C block is \( MT \) for integer \( M \geq 1 \). Further suppose the anti-aliasing filter \( H_a(j\Omega) \) is a zero-phase filter with magnitude response shown in Fig. 2 with \( \Omega_1 = 2\pi \cdot 5000 \) and \( \Omega_2 = 2\pi \cdot 19000 \) (the figure is not to scale).

\[
\begin{align*}
x_c(t) & \quad \xrightarrow{H_a(j\Omega)} \quad \xrightarrow{C/D} \quad \xrightarrow{H(e^{jw})} \quad \xrightarrow{\downarrow M} \quad \xrightarrow{D/C} \quad y(t)
\end{align*}
\]

\( T \) \hspace{2cm} \( MT \)

Figure 1: System for processing \( x_c(t) \).

\[
\begin{align*}
H_a(j\Omega)
\end{align*}
\]

\( \Omega_1 \) \hspace{2cm} \( \Omega_2 \)

Figure 2: Anti-aliasing filter spectrum.

We desire the overall system from \( x_c(t) \) to \( y(t) \) to be an ideal lowpass filter with cutoff frequency \( \Omega_c = 2\pi \cdot 4000 \).

(a) 20 points. What is the minimum sampling frequency \( f_s = \frac{1}{T} \) that can be used to achieve the desired overall response for any input? Explain.

(b) 30 points. Suppose \( f_s = 100 \text{ kHz} \). Specify discrete-time system \( H(e^{jw}) \) that achieves the desired overall response. What is the maximum value of \( M \) that can be used without affecting the desired overall response? Explain.
2. 50 points. Consider an oversampled ADC with noise shaping, modeled as shown in Fig. 3. The oversampled discrete-time signal $x[n]$ is assumed to be zero mean and stationary with variance $\sigma_x^2$. The discrete-time quantization noise $e[n]$ is assumed to be zero mean, white, stationary with variance $\sigma_e^2$, and uncorrelated with $x[n]$. The input to the system is also assumed to be band limited to $\Omega_N$ so that the output of the overall system can be written as

$$x_d[n] = x[n] + f[n]$$

where $f[n]$ is the quantization noise at the output of the system. Suppose the system that shapes the noise ($H(z)$) has magnitude response as shown in Fig. 4. Determine the signal to quantization noise ratio (SQNR) at the output of this system as a function of $\sigma_e^2$, $\sigma_x^2$, and $M$. Compare your result to the SQNR of conventional noise shaping with $H(z) = 1 - z^{-1}$.

![Diagram](image)

**Figure 3:** Oversampled ADC with noise shaping (from O & S textbook).

![Diagram](image)

**Figure 4:** Magnitude response of noise shaping filter.
1. a) Suppose \( x(t) \) is a white noise signal. After sampling, we have \( X(e^{j\omega}) \)

\[
\frac{1}{T}
\]

\( 2\pi \cdot 4000T \)

\( 2\pi \cdot 5000T \quad 2\pi \cdot 19000T \)

This point is \( 2\pi - 2\pi \cdot 19000T \)

We need \( 2\pi - 2\pi \cdot 19000T \geq 2\pi \cdot 4000T \) to avoid aliasing into the passband of our overall filter.

\[
I \geq 23000T
\]

So \( f_s \geq 23000 \) or \( T \leq \frac{1}{23000} \)

b) If \( f_s = 100 \text{ kHz} \) then after sampling we have \( X(e^{j\omega}) \)

\[
\frac{1}{T}
\]

\[-2\pi \quad 2\pi \]

We can apply an ideal DT lowpass filter here with cutoff \( \omega_c = \frac{2\pi \cdot 4}{100} \)

and the output of this filter will be

When we down sample, to avoid aliasing, we need

\[
2\pi \cdot \frac{4}{100} \cdot M \leq 2\pi - 2\pi \cdot \frac{4M}{100}
\]

\[
\frac{M}{25} \leq 1 - \frac{M}{25} \Rightarrow \frac{2M}{25} \leq 1 \Rightarrow M \leq 12
\]
2. First look at the PSD of $\hat{e}[n]$

$$P_{\hat{e}\hat{e}}(e^{jw}) = |H(e^{jw})|^2 P_{\hat{e}}(e^{jw})$$

For $-\pi \leq w \leq \pi$, we have $|H(e^{jw})| = \frac{10 \cdot |w|}{\pi}$

so

$$P_{\hat{e}\hat{e}}(e^{jw}) = \sigma_e^2 \cdot \frac{100 \cdot w^2}{\pi^2}$$

Now, when we pass this through the LPF, the quantization noise power at the output of the LPF is:

$$\sigma_v^2 = \frac{1}{2\pi} \int_{-\pi/4}^{\pi/4} \sigma_e^2 \cdot \frac{100 \cdot w^2}{\pi^2} \, dw = \frac{1}{2\pi} \cdot \sigma_e^2 \cdot \frac{100}{\pi^2} \int_{-\pi/4}^{\pi/4} w^2 \, dw$$

$$= \frac{50 \sigma_e^2}{\pi^3} \cdot \frac{2 \cdot \omega^3}{3} \Big|_{0}^{\pi/4} = \frac{50 \sigma_e^2}{\pi^3} \cdot \frac{2}{3} \cdot \frac{\pi^3}{M^3}$$

$$= \frac{100 \sigma_e^2}{3M^3} \approx 33.3 \frac{\sigma_e^2}{M^3}$$

The downsample doesn't change the power, so $\sigma_p^2 \approx \sigma_v^2$ and we have

$$SQNR = 10 \log_{10} \left( \frac{\sigma_x^2}{\sigma_p^2} \right) = 10 \log_{10} \left( \frac{\sigma_x^2 \cdot 3M^3}{100 \cdot \sigma_e^2} \right)$$

$$= 10 \log_{10} \left( \frac{3 \sigma_x^2}{100 \sigma_e^2} \right) + 30 \log_{10} (M)$$

As was the case with the usual noise shaping $H(z)=1-z^{-1}$, we have a $30 \log_{10} (M)$ term. The quantization noise variance in that case, however, was

$$\sigma_v^2 = \frac{2 \pi^2}{6M^3} \cdot \sigma_e^2 \approx 3.3 \frac{\sigma_e^2}{M^3}$$

which is $\approx 10$ times better than the system considered here.