ECE504: Lecture 12

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02-Dec-2008

Lecture 12 Major Topics

We are now in Part III of ECE504:

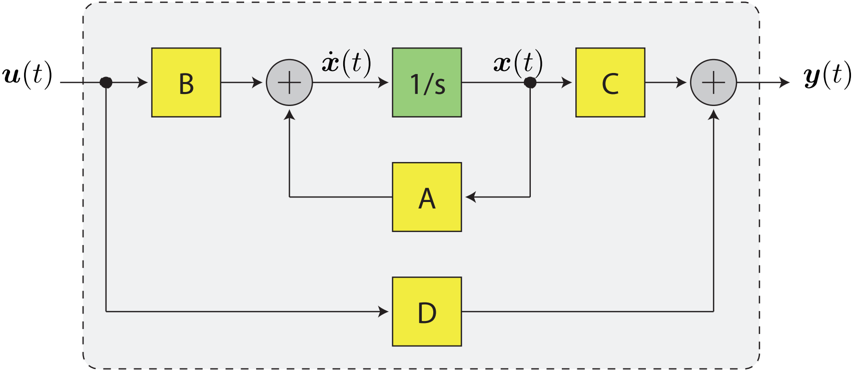
design and control of systems

Today, we will cover:

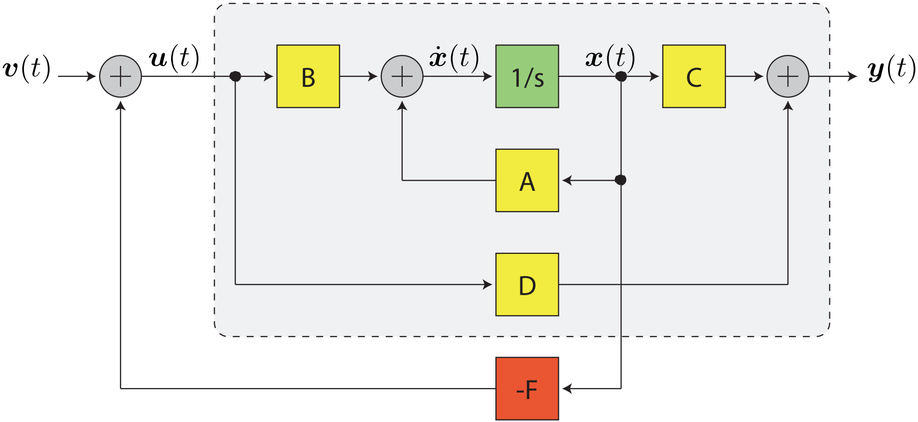
- How to control a system when you can't directly measure the current state.
- State estimators.
- State feedback with estimated states.

You should be reading Chen Chapter 8 now.

Uncontrolled System



System With State Feedback (Controlled)



A Dubious Assumption?

A key assumption of state-feedback:

We assume that we can measure the current state of the system.

This may not be possible in many practical systems. What can we do?

One approach:

1. Using the same $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ as the uncontrolled system, build an "auxiliary system" with state $\boldsymbol{w}(t)$.
2. Set $\boldsymbol{w}(0) = \boldsymbol{x}(0)$, then

$$\boldsymbol{w}(t) = \exp\{t\boldsymbol{A}\}\boldsymbol{w}(0) + \int_0^t \exp\{(t-\tau)\boldsymbol{A}\}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau = \boldsymbol{x}(t)$$

Since $\boldsymbol{w}(t) = \boldsymbol{x}(t)$ for all $t \in \mathbb{R}$ (and we built this system, so we have access to its state), we can just use $\boldsymbol{w}(t)$ for state feedback, right?

A More Realistic Approach

One big problem: we can't realistically expect to measure $\boldsymbol{x}(0)$ if we can't measure $\boldsymbol{x}(t)$.

Under the assumption that we know $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ but we don't know $\boldsymbol{x}(t)$ for any t, we can build a state estimator instead:

1. Using the same $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ as the uncontrolled system, build an "auxiliary system" with state $\boldsymbol{w}(t)$.
2. Our auxiliary system observes the output $\boldsymbol{y}(t)$ of the uncontrolled system.
3. We design our auxiliary system so that its state $\boldsymbol{w}(t)$ "tracks" the state of the uncontrolled (or controlled) system $\boldsymbol{x}(t)$ in some sense.
4. We built this system, so we have access to its state.

This auxiliary system is called a "state estimator" or a "state observer".

State Estimator Design (CT-LTI Systems)

If we just guess at $\boldsymbol{w}(0)$, the auxiliary system's state is simply

$$\boldsymbol{w}(t) = \exp\{t\boldsymbol{A}\}\boldsymbol{w}(0) + \int_0^t \exp\{(t-\tau)\boldsymbol{A}\}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau.$$

We would like $\|\boldsymbol{w}(t) - \boldsymbol{x}(t)\| \to 0$, but there is no guarantee that this will happen.

Idea: Use the output $\boldsymbol{y}(t)$ of the uncontrolled system to drive the state of our auxiliary system $\boldsymbol{w}(t)$ to the state of the actual system $\boldsymbol{x}(t)$.

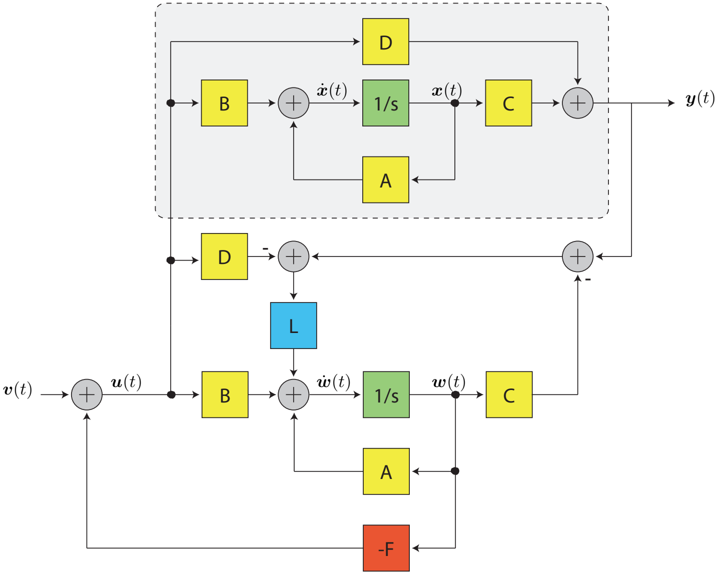
Note that if $\boldsymbol{w}(t) \approx \boldsymbol{x}(t)$, then $\boldsymbol{C}\boldsymbol{w}(t) + \boldsymbol{D}\boldsymbol{u}(t) \approx \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) = \boldsymbol{y}(t)$.

Let $\boldsymbol{L} \in \mathbb{R}^{n\times q}$ and

$$\boxed{\dot{\boldsymbol{w}}(t) = \boldsymbol{A}\boldsymbol{w}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \underbrace{\boldsymbol{L}\Big(\boldsymbol{y}(t) - \boldsymbol{C}\boldsymbol{w}(t) - \boldsymbol{D}\boldsymbol{u}(t)\Big)}_{\text{correction term}}}$$

It should be clear that the correction term here is small if $\boldsymbol{w}(t) \approx \boldsymbol{x}(t)$.

System With State Feedback Using Estimated State



State Feedback Using Estimated State

To see why this can work, look at the two state update equations

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \dot{\boldsymbol{w}}(t) &= \boldsymbol{A}\boldsymbol{w}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}\Big(\boldsymbol{y}(t) - \boldsymbol{C}\boldsymbol{w}(t) - \boldsymbol{D}\boldsymbol{u}(t)\Big)\end{aligned}$$

Take the difference to get

$$\frac{d}{dt}(\boldsymbol{x}(t) - \boldsymbol{w}(t)) = \boldsymbol{A}(\boldsymbol{x}(t) - \boldsymbol{w}(t)) - \boldsymbol{L}\Big(\boldsymbol{y}(t) - \boldsymbol{C}\boldsymbol{w}(t) - \boldsymbol{D}\boldsymbol{u}(t)\Big)$$

But $\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$. So

$$\frac{d}{dt}(\boldsymbol{x}(t) - \boldsymbol{w}(t)) = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})(\boldsymbol{x}(t) - \boldsymbol{w}(t))$$

or, more simply

$$\dot{\boldsymbol{z}}(t) = \tilde{\boldsymbol{A}}\boldsymbol{z}(t).$$

Note that this is a CT-LTI system with no input. Under what conditions on $\tilde{\boldsymbol{A}}$ will $\|\boldsymbol{z}(t)\| \to 0$ for any $\boldsymbol{z}(0)$?

State Feedback Using Estimated State

Implications

1. With this approach, we don't need to know $\boldsymbol{x}(t)$ but we do need to know $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ of the uncontrolled system.
2. We can make any guess we want for $\boldsymbol{w}(0)$.
3. Even if our guess $\boldsymbol{w}(0)$ is nowhere near $\boldsymbol{x}(0)$, if $\boldsymbol{A} - \boldsymbol{LC}$ is Hurwitz then $\boldsymbol{w}(t)$ will track $\boldsymbol{x}(t)$ closely after an initial transient.
4. The duration of this transient depends on the real part of the eigenvalues of $\boldsymbol{A} - \boldsymbol{LC}$.

Theorem

Given an uncontrolled CT-LTI system with state-space representation $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}\}$, if $\boldsymbol{A}$ and $\boldsymbol{C}$ are such that the system is observable, then we can place the eigenvalues of $\tilde{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{LC}$ arbitrarily.

The proof is based on Wonham's eigenvalue placement theorem and standard duality arguments between observability and reachability.

Putting it All Together (1 of 2)

Uncontrolled system:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)\end{aligned}$$

This system may be unstable or have undesired dynamics. We can often change this behavior with state feedback at the input of the system, i.e. $\boldsymbol{u}(t) = -\boldsymbol{F}\boldsymbol{x}(t) + \boldsymbol{v}(t)$, to get

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F})\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{v}(t)\\ \boldsymbol{y}(t) &= (\boldsymbol{C} - \boldsymbol{D}\boldsymbol{F})\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{v}(t)\end{aligned}$$

- $\boldsymbol{F}$ depends on $\boldsymbol{A}$, $\boldsymbol{B}$, and the desired eigenvalues of the system.
- If the system is reachable, then the system eigenvalues (the eigenvalues of $(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F})$) can be placed anywhere.
- Implementation requires perfect measurements of the state (may be unrealistic).

Putting it All Together (2 of 2)

If we can't directly measure the state of the uncontrolled system, we can build an auxiliary system connected to the output of the actual system

$$\dot{\boldsymbol{w}}(t) = \boldsymbol{A}\boldsymbol{w}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}\Big(\boldsymbol{y}(t) - \boldsymbol{C}\boldsymbol{w}(t) - \boldsymbol{D}\boldsymbol{u}(t)\Big)$$

and do state feedback using the estimated state
i.e. $\boldsymbol{u}(t) = -\boldsymbol{F}\boldsymbol{w}(t) + \boldsymbol{v}(t)$, to get

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) - \boldsymbol{B}\boldsymbol{F}\boldsymbol{w}(t) + \boldsymbol{B}\boldsymbol{v}(t) \\
\boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) - \boldsymbol{D}\boldsymbol{F}\boldsymbol{w}(t) + \boldsymbol{D}\boldsymbol{v}(t)
\end{aligned}$$

- $\boldsymbol{F}$ depends on $\boldsymbol{A}$, $\boldsymbol{B}$, and the desired eigenvalues of the system.
- $\boldsymbol{L}$ depends on $\boldsymbol{A}$, $\boldsymbol{C}$, and the desired eigenvalues of the estimator.
- If the system is reachable, then the system eigenvalues (the eigenvalues of $(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F})$) can be placed anywhere.
- If the system is observable, then the state estimator eigenvalues (the eigenvalues of $(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})$) can be placed anywhere.

Transfer Functions

Look at the transfer function of a system with state feedback using the actual state

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= (\boldsymbol{A}-\boldsymbol{B}\boldsymbol{F})\boldsymbol{x}(t)+\boldsymbol{B}\boldsymbol{v}(t)\\ \boldsymbol{y}(t) &= (\boldsymbol{C}-\boldsymbol{D}\boldsymbol{F})\boldsymbol{x}(t)+\boldsymbol{D}\boldsymbol{v}(t)\end{aligned}$$

We know that

$$\hat{\boldsymbol{g}}(s)=\frac{\hat{\boldsymbol{y}}(s)}{\hat{\boldsymbol{v}}(s)}=(\boldsymbol{C}-\boldsymbol{D}\boldsymbol{F})(s\boldsymbol{I}_n-(\boldsymbol{A}-\boldsymbol{B}\boldsymbol{F}))^{-1}\boldsymbol{B}+\boldsymbol{D}$$

Now look at the transfer function of a system with state feedback using the estimated state...

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t)-\boldsymbol{B}\boldsymbol{F}\boldsymbol{w}(t)+\boldsymbol{B}\boldsymbol{v}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t)-\boldsymbol{D}\boldsymbol{F}\boldsymbol{w}(t)+\boldsymbol{D}\boldsymbol{v}(t)\end{aligned}$$

Need to do a bit more work in this case...

Examples

Conclusions

1. This concludes Part III of ECE504: design and control of systems.
2. All of our analysis was for CT-LTI systems, but these ideas also apply directly to DT-LTI systems.
3. Qualitative properties of systems are very important in control:
   - Reachability
   - Observability
4. Lower dimensional problems ($n = 2$ or $n = 3$ with $p = q = 1$) can usually be solved by hand.
5. Higher dimensional problems require Matlab.
6. Lots more interesting control topics:
   - Regulation and Tracking (see Chen section 8.3)
   - Disturbance rejection (see Chen section 8.3.1)
   - Reduced-dimensional state estimators (see Chen section 8.4.1)
   - Optimal control (find an input to minimize a cost function, e.g. energy)