ECE504: Lecture 2

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Lecture 2 Major Topics

We are still in Part I of ECE504: **Mathematical description of systems**

$$\text{model} \longrightarrow \text{mathematical description}$$

You should be reading Chen chapters 2-3 now.

1. Advantages and disadvantages of different mathematical descriptions
2. Transfer functions review
3. Relationships between mathematical descriptions

Preliminary Definition: Relaxed Systems

Definition

A system is said to be "relaxed" at time $t = t_0$ if the output $y(t)$ for all $t \geq t_0$ is excited exclusively by the input $u(t)$ for $t \geq t_0$.

Input-Output Description: Capabilities and Limitations

Example:

$$ay(t) + \frac{b\dot{y}(t)}{c\ddot{y}(t)} = du(t) + e\dot{u}(t)$$

+ Can describe memoryless, lumped, or distributed systems.
+ Can describe causal or non-causal systems.
+ Can describe linear or non-linear systems.
+ Can describe time-invariant or time-varying systems.
+ Can describe relaxed or non-relaxed systems (non-zero initial conditions).

— No explicit access to internal behavior of systems, e.g. doesn't directly to apply to systems like "sharks and sardines".

— Difficult to analyze directly (differential equations).

State-Space Description: Capabilities and Limitations

Example:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)\end{aligned}$$

— Can't describe distributed systems. Only memoryless or lumped systems.
— Can't describe non-causal systems. Only causal systems.
+ Can describe linear or non-linear systems (the example is linear).
+ Can describe time-invariant or time-varying systems.
+ Can describe relaxed or non-relaxed systems (non-zero initial conditions).
+ Explicit description of internal system behavior, e.g. we can determine the stability of signals internal to a system.
+ Abundance of analysis techniques. Linear state-space descriptions are analyzed with **linear algebra**, not calculus.

Transfer Function Description: Capabilities and Limitations

Example:

$$\hat{g}(s) = \frac{as^2 + bs + c}{ds^3 + 1}$$

+ Can describe memoryless, lumped, and some distributed systems.
— Can't describe non-causal systems. Only causal systems.
— Can't describe non-linear systems. Only linear systems.
— Can't describe time-varying systems. Only time-invariant systems.
— No explicit access to internal behavior of systems.
— Can't describe systems with non-zero initial conditions. Implicitly assumes that system is relaxed.
+ Abundance of analysis techniques. Systems are usually analyzed with **basic algebra**, not calculus.

Impulse Response Description: Capabilities and Limitations

- Recall that the **impulse response** of a system is the output of the system given an input $u(t) = \delta(t)$ and relaxed initial conditions.
- Example:

$$g(t) = \begin{cases} \beta e^{-\alpha t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

- What are the capabilities and limitations of the impulse-response description?

Impulse Response Matrix

Suppose you have a linear system with p inputs and q outputs. Rather than a simple impulse response function, you now need an impulse response matrix:

$$\boldsymbol{G}(t,\tau) = \begin{bmatrix} g_{11}(t,\tau) & \dots & g_{1p}(t,\tau) \\ \vdots & & \vdots \\ g_{q1}(t,\tau) & \dots & g_{qp}(t,\tau) \end{bmatrix}$$

where $g_{k\ell}(t,\tau)$ is the response at the k^{th} output from an impulse at the ℓ^{th} input at time τ. The vector output can be computed as

$$\boldsymbol{y}(t) = \int_{-\infty}^{\infty} \boldsymbol{G}(t,\tau)\boldsymbol{u}(\tau)\,d\tau$$

- Sanity check: What are the dimensions of everything here?
- How do we integrate the vector $\boldsymbol{G}(t,\tau)\boldsymbol{u}(\tau)$?

The One-Sided Laplace Transform

- Suppose $\boldsymbol{f}(t) : \mathbf{R}_+ \mapsto \mathbf{R}^{q \times p}$ is a matrix valued function of $t \geq 0$, i.e.

$$\boldsymbol{f}(t) = \begin{bmatrix} f_{11}(t) & \dots & f_{1p}(t) \\ \vdots & & \vdots \\ f_{q1}(t) & \dots & f_{qp}(t) \end{bmatrix}$$

- Define the Laplace transform of $\boldsymbol{f}(t)$

$$\hat{\boldsymbol{f}}(s) = \int_0^\infty e^{-st} \boldsymbol{f}(t)\, dt$$

- The integral of a matrix is done element by element.
- The notation here is consistent with your textbook:

$$\begin{aligned} \hat{\boldsymbol{f}}(s) &= \mathcal{L}\left[\boldsymbol{f}(t)\right] \\ \boldsymbol{f}(t) &= \mathcal{L}^{-1}\left[\hat{\boldsymbol{f}}(s)\right] \\ \boldsymbol{f}(t) &\leftrightarrow \hat{\boldsymbol{f}}(s) \end{aligned}$$

Convergence of the One-Sided Laplace Transform

$$\hat{\boldsymbol{f}}(s) = \int_0^\infty e^{-st} \boldsymbol{f}(t)\, dt$$

Let $\Lambda_{k\ell} \subseteq \mathbb{R}$ be the set of all $\sigma \in \mathbb{R}$ such that $\int_0^\infty |f_{k\ell}(t)| e^{-\sigma t}\, dt < \infty$.

If

$$\Lambda = \bigcap_{k=1}^{q} \bigcap_{\ell=1}^{p} \Lambda_{k\ell} = \varnothing$$

then the one-sided Laplace transform doesn't exist. Otherwise, let $a = \min \Lambda$. The set of complex numbers such that $\{s \in \mathbb{C} : \mathrm{Re}[s] > a\}$ is called the "region of absolute convergence" of the Laplace transform of the matrix valued function $\boldsymbol{f}(t)$.

The Inverse Laplace Transform

$$\boldsymbol{f}(t) = \frac{1}{2\pi j}\int_{\sigma-j\infty}^{\sigma+j\infty} e^{st}\hat{\boldsymbol{f}}(s)\,ds$$

where $j := \sqrt{-1}$ and σ is any point in the region of absolute convergence of $\hat{\boldsymbol{f}}(s)$.

- This is *complex* integration on a path in the complex plane.
- In general, this integral is usually not easy to compute.
- Whenever possible, use tables instead.

Continuous-Time Transfer Function

Definition

Given a causal, linear, time-invariant system with p input terminals, q output terminals, and relaxed initial conditions at time $t = 0$

$$\left.\begin{array}{l} \boldsymbol{x}(0) = 0 \\ \boldsymbol{u}(t),\ t \geq 0 \end{array}\right\} \rightarrow \boldsymbol{y}(t),\ t \geq 0$$

then the transfer function matrix is defined as

$$\hat{\boldsymbol{g}}(s) \quad := \quad \begin{bmatrix} \hat{g}_{11}(s) & \dots & \hat{g}_{1p}(s) \\ \vdots & & \vdots \\ \hat{g}_{q1}(s) & \dots & \hat{g}_{qp}(s) \end{bmatrix} \text{ where } \hat{g}_{k\ell}(s) := \frac{\hat{y}_k(s)}{\hat{u}_\ell(s)}$$

for $k = 1, \dots, q$ and $\ell = 1, \dots, p$.

Continuous-Time Transfer Function: Remarks

- By the relaxed assumption, the transfer function describes the **zero-state response** of the system.
- Since the transfer function must be the same for any input/output combination, most textbooks (including Chen) define it as the Laplace transform of the impulse response of the system.
- What is the Laplace transform of $\delta(t)$?
- Hence, given a causal, linear, time-invariant system with p input terminals, q output terminals, and relaxed initial conditions

$$\left.\begin{array}{l} \boldsymbol{x}(0) = 0 \\ u_\ell(t) = \delta(t) \text{ and } u_m(t) = 0 \text{ for all } m \neq \ell,\ t \geq 0 \end{array}\right\} \rightarrow \boldsymbol{y}(t),\ t \geq 0$$

then

$$\begin{bmatrix} \hat{g}_{1\ell}(s) \\ \vdots \\ \hat{g}_{q\ell}(s) \end{bmatrix} = \begin{bmatrix} \frac{\hat{y}_1(s)}{\hat{u}_\ell(s)} \\ \vdots \\ \frac{\hat{y}_q(s)}{\hat{u}_\ell(s)} \end{bmatrix} = \begin{bmatrix} \hat{y}_1(s) \\ \vdots \\ \hat{y}_q(s) \end{bmatrix}$$

Rational Transfer Functions and Degree

Theorem (stated as a fact, Chen p.14)

If a continuous-time, linear, time-invariant system is lumped, each transfer function in the transfer function matrix is a rational function of s.

The proof for this theorem can be found in some of the other textbooks mentioned in Lecture 1. In any case, this result means that

$$\hat{g}_{k\ell}(s) = \frac{N(s)}{D(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

where $N(s)$ and $D(s)$ are polynomials of s.

Definition

The degree of a polynomial in s is the highest power of s in the polynomial.

Example: $\deg(0 \cdot s^4 + 2 \cdot s^3 + 6) =$ ______________.

The Three Most Important Laplace Transforms: #1

$$\boxed{e^{at} \leftrightarrow \frac{1}{s-a}}$$

- Easy to show directly by integrating according to the definition.
- Application: For a continuous-time, linear, time-invariant, lumped system, we have

  $$\hat{g}(s) = \frac{b_m s^m + b_{m-1} s^{m-1} + \cdots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \cdots + a_1 s + a_0}$$

  Compute partial fraction expansion ...

  $$\hat{g}(s) = \frac{c_1}{s-d_1} + \cdots + \frac{c_n}{s-d_n}$$

  Then what can we say about $g(t)$?
- Note that this is slightly more complicated if the roots of the denominator are repeated.

The Three Most Important Laplace Transforms: #2

Notation: $g^{(n)}(t) := \frac{dg^n(t)}{dt^n}$.

$$\boxed{g^{(n)}(t) \leftrightarrow s^n \hat{g}(s) - s^{n-1} g(0) - s^{n-2} \dot{g}(0) - \cdots - g^{(n-1)}(0)}$$

- $n = 1$ case is especially useful: $\dot{g}(t) \leftrightarrow s\hat{g}(s) - g(0)$.
- General relationship can be shown inductively using the definition and integration by parts.

The Three Most Important Laplace Transforms: #2

Application #1: Given the input-output differential equation description of a continuous-time, linear, time-invariant, lumped system, we can easily compute the transfer function.

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 \dot{y}(t) + a_0 y(t) =$$
$$b_m u^{(m)}(t) + b_{m-1} u^{(m-1)}(t) + \cdots + b_1 \dot{u}(t) + b_0 u(t)$$

The Three Most Important Laplace Transforms: #2

Application #2: Given the state space description of a continuous-time, linear, time-invariant, lumped system, we can easily compute the transfer function.

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)\end{aligned}$$

The Three Most Important Laplace Transforms: #3

Notation (convolution assuming a causal relaxed system):

$$f(t) * g(t) := \int_0^t f(t-\tau)g(\tau)\,d\tau = \int_0^t f(\tau)g(t-\tau)\,d\tau.$$

It isn't too hard to show that

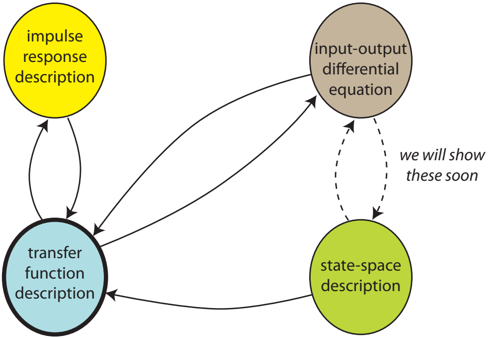
$$\boxed{f(t) * g(t) \leftrightarrow \hat{f}(s)\hat{g}(s)}$$

- Application #1: By definition of the transfer function of a linear, time-invariant, causal system, $\hat{y}(s) = \hat{g}(s)\hat{u}(s)$. This implies that

$$y(t) = \int_0^t g(t-\tau)u(\tau)\,d\tau = \int_0^t g(\tau)u(t-\tau)\,d\tau.$$

- Application #2: When $u(t) = \delta(t)$, you can show that $y(t) = g(t)$. Hence $g(t) = \mathcal{L}^{-1}\left[\hat{g}(s)\right]$ is the impulse response of the system.

What We Know: Moving Between System Descriptions



- You've seen transfer functions in an undergraduate course (hopefully).
- A transfer function (matrix) describes the zero-state response of a causal LTI system (relaxed initial conditions at time t_0).
- Transfer functions can be used to represent some distributed systems, but these systems can't be represented with a state-space description.

Moving Between SS and I/O Descriptions

For now, we will focus on the linear, time-invariant case.

Theorem

Given a continuous-time input-output differential equation

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 \dot{y}(t) + a_0 y(t) = \\ b_m u^{(m)}(t) + b_{m-1} u^{(m-1)}(t) + \cdots + b_1 \dot{u}(t) + b_0 u(t)$$

a state-space description of this systems exists if and only if $m \leq n < \infty$.

- Note that this theorem states "if and only if". What does this mean?
- How can we prove this theorem?

An Easy Way to Go From an LTI I/O to SS Description

To prove the "if" part of the theorem, we are going to show that given a continuous-time, linear, time-invariant, lumped system with input-ouput differential equation

$$\begin{aligned} a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 \dot{y}(t) + a_0 y(t) = \\ b_m u^{(m)}(t) + b_{m-1} u^{(m-1)}(t) + \cdots + b_1 \dot{u}(t) + b_0 u(t) \end{aligned}$$

we can always write a linear, time-invarant state-space description

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) \end{aligned}$$

of the system.

An Easy Way to Go From an LTI I/O to SS Description

First derive a state dynamic equation...

An Easy Way to Go From an LTI I/O to SS Description

Now derive the output equation...

- Case I: $m < n < \infty$.
- Case II: $m = n < \infty$.

Hence, we have proved the "if" part of the theorem.

Going from an LTI SS to I/O Description

To prove the "only if" part of the theorem, we are going to show that a linear, time-invaraint state-space description

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)\end{aligned}$$

can always be written as an I/O differential equation

$$a_n y^{(n)}(t) + a_{n-1} y^{(n-1)}(t) + \cdots + a_1 \dot{y}(t) + a_0 y(t) = \\ b_m u^{(m)}(t) + b_{m-1} u^{(m-1)}(t) + \cdots + b_1 \dot{u}(t) + b_0 u(t)$$

with $m \leq n < \infty$.

To do this, however, we are going to need to learn a bit of linear algebra:

- The determinant of a square matrix.
- The adjoint of a square matrix.
- The matrix inverse.
- The matrix inverse in terms of the determinant and the adjoint.

The Determinant of a Square Matrix

Given $\boldsymbol{W} \in \mathbb{R}^{n \times n}$, let $\boldsymbol{M}_{ij}$ be the $(n-1) \times (n-1)$ square matrix formed by deleting the i^{th} row and the j^{th} column of $\boldsymbol{W}$.

Definition

Given $\boldsymbol{W} \in \mathbb{R}^{n \times n}$, the determinant is defined recursively as

$$\det[\boldsymbol{W}] = \sum_{i=1}^{n} w_{ij} (-1)^{i+j} \det[\boldsymbol{M}_{ij}]$$

for any $j \in \{1, \dots, n\}$ and where the determinant of any scalar $x \in \mathbb{R}$ is simply $\det[x] = x$.

Remark: $c_{ij} := (-1)^{i+j} \det[\boldsymbol{M}_{ij}]$ is called the ij^{th} cofactor of $\boldsymbol{W}$.

Examples...

The Adjoint of a Square Matrix

Definition

The adjoint $\boldsymbol{J} \in \mathbb{R}^{n \times n}$ of the matrix $\boldsymbol{W} \in \mathbb{R}^{n \times n}$ is defined as

$$\boldsymbol{J} = \operatorname{adj}(\boldsymbol{W}) = \begin{bmatrix} c_{11} & \dots & c_{1n} \\ \vdots & & \vdots \\ c_{n1} & \dots & c_{nn} \end{bmatrix}^\top$$

where c_{ij} the ij^{th} cofactor of $\boldsymbol{W}$.

Remarks:

- Note the transpose.
- The adjoint of any scalar $x \in \mathbb{R}$ is simply $\operatorname{adj}[x] = 1$.

Examples...

The Matrix Inverse in Terms of the Det. and the Adjoint

Theorem (Hoffman and Kunze, Linear Algebra, 2nd Edition, p.160)

Let $\boldsymbol{A}$ be an $n \times n$ matrix. When $\boldsymbol{A}$ is invertible, the unique inverse for $\boldsymbol{A}$ is

$$\boldsymbol{A}^{-1} = \frac{\mathrm{adj}(\boldsymbol{A})}{\det(\boldsymbol{A})}$$

Remarks:

- See any decent linear algebra textbook for the proof.
- Note that $\boldsymbol{A}$ is invertible if and only if $\det(\boldsymbol{A}) \neq 0$.
- This is not how matrix inverses are actually computed in programs like Matlab and Octave (there are more computationally efficient ways to get the same answer). Nevertheless, we can use this result in our proof of the "only if" part.

Going from an LTI SS to I/O Description

Back to the "only if" part of the proof. Recall that we can go from a linear, time-invariant state-space description to a transfer function by computing

$$\hat{\boldsymbol{g}}(s) = \boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}+\boldsymbol{D} = \frac{N(s)}{D(s)}$$

Using what we now know about matrix inverses, we can write

$$\boldsymbol{C}(s\boldsymbol{I}-\boldsymbol{A})^{-1}\boldsymbol{B}+\boldsymbol{D} = \frac{\boldsymbol{C}\mathrm{adj}(s\boldsymbol{I}-\boldsymbol{A})\boldsymbol{B}}{\det(s\boldsymbol{I}-\boldsymbol{A})}+\boldsymbol{D} = \frac{\tilde{N}(s)}{D(s)}+\boldsymbol{D}$$

- What can you say about the degree of $D(s)$?
- What can you say about the degree of $\tilde{N}(s)$? Recall that the elements of $\boldsymbol{C}$ and $\boldsymbol{B}$ are constants (not functions of s).
- But what about $\boldsymbol{D}$?

Conclusions

- Capabilities and limitations of different mathematical descriptions of systems.
- Continuous time transfer function review.
- Linear, time-invariant state-space $\rightarrow$ transfer function relationship
- Linear algebra: identity matrix, matrix inverse, determinant, adjoint.
- Linear, time-invariant state-space $\leftrightarrow$ I/O differential equation relationship.