ECE504: Lecture 4

D. Richard Brown III

Worcester Polytechnic Institute

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Lecture 4 Major Topics

We are now starting Part II of ECE504: **Quantitative and qualitative analysis of systems**

mathematical description → results about behavior of system

Today:

1. Solution of state equations for discrete-time systems
2. Solution of state equations for continuous-time systems
3. Some necessary linear algebra (and calculus review)
4. Examples

You should be reading Chen Chapter 4 now. You should also read Chen 3.2-3.3 to learn about "basis", "linear independence", and solutions to linear algebraic equations like $\boldsymbol{Ax} = \boldsymbol{y}$.

Linear State-Space Description of Discrete-Time Systems

$$
\begin{aligned}
\boldsymbol{x}[k+1] &= \boldsymbol{A}[k]\boldsymbol{x}[k] + \boldsymbol{B}[k]\boldsymbol{u}[k] \\
\boldsymbol{y}[k] &= \boldsymbol{C}[k]\boldsymbol{x}[k] + \boldsymbol{D}[k]\boldsymbol{u}[k]
\end{aligned}
$$

We assume a general model with p inputs, q outputs, and n states.

Given an initial time $k_0 \in \mathbb{Z}$, an initial state $\boldsymbol{x}[k_0] \in \mathbb{R}^n$, how does the state evolve for $k = k_0 + 1, k_0 + 2, \dots$?

Solution to State Equation

Following our induction, for all $k \geq k_0$, we can write

$$\boldsymbol{x}[k] \;=\; \boldsymbol{\Phi}[k, k_0]\boldsymbol{x}[k_0] + \sum_{\ell=k_0}^{k-1} \boldsymbol{\Phi}[k, \ell+1]\boldsymbol{B}[\ell]\boldsymbol{u}[\ell]$$

where $\boldsymbol{\Phi}$ is an $n \times n$ matrix valued function with two time arguments:

$$\boldsymbol{\Phi}[k, j] \;=\; \begin{cases} \text{undefined} & k < j \\ \boldsymbol{I}_n & k = j \\ \boldsymbol{A}[k-1]\boldsymbol{A}[k-2]\cdots\boldsymbol{A}[j] & k > j \end{cases}$$

Remarks:

- $\boldsymbol{I}_n$ is the $n \times n$ identity matrix.
- The order of the product $\boldsymbol{A}[k-1]\boldsymbol{A}[k-2]\cdots\boldsymbol{A}[j]$ is important because matrices don't usually commute.
- The matrix function $\boldsymbol{\Phi} : \mathbb{Z}^2 \mapsto \mathbb{R}^{n\times n}$ is called the state transition matrix (STM) corresponding to $\boldsymbol{A}[k]$.

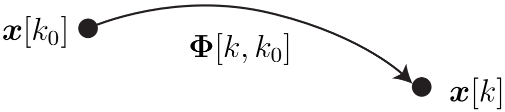
Zero-Input Response

Recall that linear systems have the nice property that we can separately analyze the zero-input response and the zero-state response.

Zero-input response: Given $\boldsymbol{u}[k] = 0$ for all $k \geq k_0$, we can write

$$\boldsymbol{x}[k] = \boldsymbol{\Phi}[k, k_0]\boldsymbol{x}[k_0]$$

The state transition matrix $\boldsymbol{\Phi}[k, k_0]$ describes how the state at time k_0 evolves to the state at time $k \geq k_0$ (in the absence of an input).



If the STM $\boldsymbol{\Phi}[k, k_0]$ is invertible, then $\boldsymbol{\Phi}^{-1}[k, k_0] = \boldsymbol{\Phi}[k_0, k]$. But there is no guarantee that it is invertible. This operation is one-way.

Zero-State Response

Zero-state response: Given $\boldsymbol{x}[k_0] = 0$, we can write

$$\boldsymbol{x}[k] = \sum_{\ell=k_0}^{k-1} \boldsymbol{\Phi}[k, \ell+1]\boldsymbol{B}[\ell]\boldsymbol{u}[\ell]$$

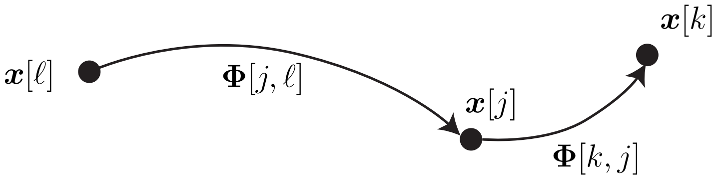
In this case, we have to compute several state transition matrices: $\boldsymbol{\Phi}[k, k_0+1], \boldsymbol{\Phi}[k, k_0+2], \ldots, \boldsymbol{\Phi}[k, k]$.

This looks like it might require a lot of computation as k gets larger. Fortunately, there are some nice properties of the state transition matrix that can ease the computational burden...

Some Basic Properties of the State Transition Matrix

1. $\mathbf{\Phi}[j,j] = \boldsymbol{I}_n$ for all $j \in \mathbb{Z}$.
2. $\mathbf{\Phi}[k+1,j] = \boldsymbol{A}[k]\mathbf{\Phi}[k,j]$ for all $k \geq j$.
3. If $\ell \leq j \leq k$, then $\mathbf{\Phi}[k,\ell] = \mathbf{\Phi}[k,j]\mathbf{\Phi}[j,\ell]$.

This last property is called the "semigroup" property. It intuitively says that the transition from $\boldsymbol{x}[\ell]$ to $\boldsymbol{x}[k]$ is the same as the transition from $\boldsymbol{x}[\ell]$ to $\boldsymbol{x}[j]$ followed by the transition from $\boldsymbol{x}[j]$ to $\boldsymbol{x}[k]$.



Special Case: $\boldsymbol{A}[k] \equiv \boldsymbol{A}$ for all $k \geq k_0$

When $\boldsymbol{A}[k] \equiv \boldsymbol{A}$ for all $k \geq k_0$, the product

$$\boldsymbol{A}[k-1]\boldsymbol{A}[k-2]\cdots\boldsymbol{A}[j] \quad = \quad \boldsymbol{A}\boldsymbol{A}\cdots\boldsymbol{A}$$

How many $\boldsymbol{A}$'s are involved in this product? ______

Hence, when $\boldsymbol{A}[k] \equiv \boldsymbol{A}$ for all $k \geq k_0$, the state transition matrix can be written as

$$\boldsymbol{\Phi}[k,j] \quad = \quad \begin{cases} \text{undefined} & k < j \\ \boldsymbol{I}_n & k = j \\ \boldsymbol{A}^{k-j} & k > j. \end{cases}$$

In this case, the solution to the DT state-update difference equation is

$$\boldsymbol{x}[k] = \boldsymbol{A}^{k-k_0}\boldsymbol{x}[k_0] + \sum_{\ell=k_0}^{k-1} \boldsymbol{A}^{k-\ell-1}\boldsymbol{B}[\ell]\boldsymbol{u}[\ell]$$

for all $k \geq k_0$.

Discrete-Time Output Solution

For all $k \geq k_0$, we can just plug our solution to the state equation into our state-space output equation to get

$$\boldsymbol{y}[k] \;=\; \underbrace{\boldsymbol{C}[k]\boldsymbol{\Phi}[k,k_0]\boldsymbol{x}[k_0]}_{\text{zero-input response}} + \underbrace{\boldsymbol{C}[k]\sum_{\ell=k_0}^{k-1}\boldsymbol{\Phi}[k,\ell+1]\boldsymbol{B}[\ell]\boldsymbol{u}[\ell] + \boldsymbol{D}[k]\boldsymbol{u}[k]}_{\text{zero-state response}}$$

If the system is time-invariant, then we can write

$$\boldsymbol{y}[k] = \underbrace{\boldsymbol{C}\boldsymbol{A}^{k-k_0}\boldsymbol{x}[k_0]}_{\text{zero-input response}} + \underbrace{\boldsymbol{C}\sum_{\ell=k_0}^{k-1}\boldsymbol{A}^{k-\ell-1}\boldsymbol{B}\boldsymbol{u}[\ell] + \boldsymbol{D}\boldsymbol{u}[k]}_{\text{zero-state response}}$$

Remarks on Discrete-Time State-Space Solutions

For causal, linear, lumped discrete-time systems with p input terminals, q output terminals, and n states, we have shown that, given $\boldsymbol{x}[k_0]$ and $\boldsymbol{u}[k]$ for all $k \geq k_0$, **there exists a unique solution** to the discrete-time state-update difference equation:

$$\boldsymbol{x}[k] \quad = \quad \boldsymbol{\Phi}[k, k_0]\boldsymbol{x}[k_0] + \sum_{\ell=k_0}^{k-1} \boldsymbol{\Phi}[k, \ell+1]\boldsymbol{B}[\ell]\boldsymbol{u}[\ell]$$

for all $k \geq k_0$ with $\boldsymbol{\Phi}[k, j]$ as defined earlier.

This also implies that, given $\boldsymbol{x}[k_0]$ and $\boldsymbol{u}[k]$ for all $k \geq k_0$, there exists a unique solution to the discrete-time output equation.

Discrete-Time State-Space Example

Continuous-Time Linear Systems

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t) && (1) \\ \boldsymbol{y}(t) &= \boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{D}(t)\boldsymbol{u}(t) && (2) \end{aligned}$$

Theorem

For any $t_0 \in \mathbb{R}$, any $\boldsymbol{x}(t_0) \in \mathbb{R}^n$, and any $\boldsymbol{u}(t) \in \mathbb{R}^p$ for all $t \geq t_0$, there exists a unique solution $\boldsymbol{x}(t)$ for all $t \in \mathbb{R}$ to the state-update differential equation (1). It is given as

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{\Phi}(t, \tau)\boldsymbol{B}(\tau)\boldsymbol{u}(\tau)\, d\tau \quad t \in \mathbb{R}$$

where $\boldsymbol{\Phi}(t, s) : \mathbb{R}^2 \mapsto \mathbb{R}^{n \times n}$ is the unique function satisfying

$$\frac{d}{dt}\boldsymbol{\Phi}(t, s) = \boldsymbol{A}(t)\boldsymbol{\Phi}(t, s) \text{ with } \boldsymbol{\Phi}(s, s) = \boldsymbol{I}_n.$$

Theorem Remarks

- Note that this theorem claims two things:
  1. A solution to the state-update equation always **exists**.
  2. The solution is **unique**.
- Our strategy to prove the theorem:
  1. We will first show that, given two solutions to the state-update equation, they must be identical. This establishes uniqueness.
  2. We will then establish existence constructively by giving a solution and showing that it satisfies the state-update equation.

Before doing any of this, however, we are going to need to learn some more linear algebra (and a calculus refresher)...

Euclidean Norm of a Vector

Definition

For $\boldsymbol{x} \in \mathbb{R}^n$, the Euclidean norm of $\boldsymbol{x}$ is given as

$$\|\boldsymbol{x}\| := \left(x_1^2 + \cdots + x_n^2\right)^{1/2}.$$

The Euclidean norm of vectors in $\mathbb{R}^1$, $\mathbb{R}^2$, or $\mathbb{R}^3$ is just your normal notion of distance/length.

Some useful facts (easy to show from the definition):

- $\|\boldsymbol{x}\|^2 = \boldsymbol{x}^\top \boldsymbol{x}$.
- $\|\alpha \boldsymbol{x}\| = |\alpha| \|\boldsymbol{x}\|$ for any α in $\mathbb{R}$.
- $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$ for any $\boldsymbol{x} \in \mathbb{R}^n$ and any $\boldsymbol{y} \in \mathbb{R}^n$. This is often called the triangle inequality.

Induced Euclidean Norm of a Matrix

Definition

For $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, the induced Euclidean norm of the matrix $\boldsymbol{A}$ is given as

$$\|\boldsymbol{A}\| \quad := \quad \max_{\boldsymbol{x} \in \mathbb{R}^n \text{ and } \|\boldsymbol{x}\|=1} \|\boldsymbol{A}\boldsymbol{x}\|.$$

- The set of vectors $\boldsymbol{x}$ where $\|\boldsymbol{x}\| = 1$ is a unit-sphere in $\mathbb{R}^n$.
- The induced Euclidean norm of $\boldsymbol{A}$ is the maximum value of $\|\boldsymbol{A}\boldsymbol{x}\|$ as $\boldsymbol{x}$ ranges over all points on this unit-sphere.
- Intuitively, $\|\boldsymbol{A}\|$ gives a measure of how much $\boldsymbol{A}$ can magnify the length (Euclidean norm) of a vector in $\mathbb{R}^n$.

Some useful facts (not too hard to show from the definition):

- $\|\boldsymbol{A} + \boldsymbol{B}\| \leq \|\boldsymbol{A}\| + \|\boldsymbol{B}\|$ for any $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and any $\boldsymbol{B} \in \mathbb{R}^{n \times n}$.
- $\|\boldsymbol{A}\boldsymbol{B}\| \leq \|\boldsymbol{A}\|\|\boldsymbol{B}\|$ for any $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and any $\boldsymbol{B} \in \mathbb{R}^{n \times n}$.
- $\|\boldsymbol{A}\boldsymbol{x}\| \leq \|\boldsymbol{A}\|\|\boldsymbol{x}\|$ for any $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and any $\boldsymbol{x} \in \mathbb{R}^n$.

Schwarz Inequality

Theorem

Given $\boldsymbol{x} \in \mathbb{R}^n$ *and* $\boldsymbol{y} \in \mathbb{R}^n$, *then*

$$|\boldsymbol{x}^\top \boldsymbol{y}| = |x_1 y_1 + \cdots + x_n y_n| \leq \|\boldsymbol{x}\| \|\boldsymbol{y}\|$$

Proof:

Leibniz' rule

Theorem

If $f(t,\tau)$ is continuous and all of the necessary derivatives exist, then

$$\frac{d}{dt}\int_{v(t)}^{w(t)} f(t,\tau)\,d\tau = \dot{w}(t)f(t,w(t)) - \dot{v}(t)f(t,v(t)) + \int_{v(t)}^{w(t)} \frac{d}{dt}f(t,\tau)\,d\tau$$

The proof can be found in most calculus textbooks.

Two particularly useful special cases are

$$\frac{d}{dt}\int_{a}^{t} f(\tau)\,d\tau = f(t)$$

$$\frac{d}{dt}\int_{t}^{a} f(\tau)\,d\tau = -f(t)$$

where a is not a function of t.

Back to the Theorem: Uniqueness Proof

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t)$$

We first want to show that any solution $\boldsymbol{x}(t)$ to this state-update differential equation must be unique.

To show this, suppose we had two solutions to the state-update differential equation, $\boldsymbol{x}_1(t)$ and $\boldsymbol{x}_2(t)$ for $t \in [s_1, t_1]$, both of which satisfy the initial condition $\boldsymbol{x}_1(t_0) = \boldsymbol{x}_2(t_0) = \boldsymbol{x}(t_0)$. Let's prove that $\boldsymbol{x}_1(t)$ must be identical to $\boldsymbol{x}_2(t)$...

Theorem: Existence Proof Warmup

We now know that, if a solution to the state-update DE exists, it must be unique. We now need to show that a solution always exists.

To develop some intuition, let's first assume that everything is scalar, i.e. $p = q = n = 1$. Our state update equation becomes

$$\dot{x}(t) \;\;=\;\; a(t)x(t) + b(t)u(t)$$

Let

$$\phi(t,s) \;\;:=\;\; \exp\left\{\int_s^t a(\tau)\,d\tau\right\}$$

What is $\phi(s,s)$?

What is $\frac{d}{dt}\phi(t,s)$?

Theorem: Existence Proof Warmup

Note that $\phi(t,s) = \exp\left\{\int_s^t a(\tau)\,d\tau\right\}$ always exists and satisfies its own differential equation:

$$\frac{d}{dt}\phi(t,s) = a(t)\phi(t,s) \text{ with } \phi(s,s) = 1.$$

Now lets try the following solution to the scalar state-update differential equation with initial state condition $x(t_0)$:

$$x(t) \;=\; \phi(t,t_0)x(t_0) + \int_{t_0}^{t} \phi(t,\tau)b(\tau)u(\tau)\,d\tau \qquad \forall t \in \mathbb{R}$$

To see that this solution is valid, we should confirm two things:

1. Does our solution satisfy the initial condition requirement of the scalar state-update DE?
2. Does our solution really solve the scalar state-update DE?

Theorem: Existence Proof

For the general (non-scalar) case, we propose the solution

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{\Phi}(t, \tau)\boldsymbol{B}(\tau)\boldsymbol{u}(\tau)\, d\tau \tag{3}$$

where the state transition matrix satisfies the matrix differential equation

$$\frac{d}{dt}\boldsymbol{\Phi}(t, s) = \boldsymbol{A}(t)\boldsymbol{\Phi}(t, s) \text{ with } \boldsymbol{\Phi}(s, s) = \boldsymbol{I}_n. \tag{4}$$

To complete the existence proof, we need to:

1. Show that (3) with $\boldsymbol{\Phi}$ defined according to (4) satisfies the initial condition requirement of the state-update DE.
2. Show that (3) with $\boldsymbol{\Phi}$ defined according to (4) is indeed a solution to the state-update DE.
3. Show that there always exists a solution to the matrix DE (4).

Remarks on the CT State-Transition Matrix $\mathbf{\Phi}(t, s)$

1. Computation of $\mathbf{\Phi}(t, s)$ is almost always difficult.
2. The Peano-Baker series is only one way to compute $\mathbf{\Phi}(t, s)$. Other (perhaps better?) ways:
   - Directly solve the matrix state-update differential equations (not always possible)
   - Fundamental matrix method (see Chen 4.5)
   - Other methods...
3. Question: Is it possible that different methods for computing the STM will lead to different $\mathbf{\Phi}(t, s)$?
4. Unlike the DT-STM $\mathbf{\Phi}[k, j]$, the CT-STM $\mathbf{\Phi}(t, s)$ is defined for any $(t, s) \in \mathbb{R}^2$. This means that we can specify an initial state $\boldsymbol{x}(t_0)$ and compute the system response at times **prior** to t_0.
5. $\mathbf{\Phi}(t, s)$ possesses the semi-group property, i.e.

$$\mathbf{\Phi}(t, \tau) = \mathbf{\Phi}(t, s)\mathbf{\Phi}(s, \tau)$$

for any $(t, \tau, s) \in \mathbb{R}^3$.

Important Special Case: $A(t) \equiv A$

When $\boldsymbol{A}(t) \equiv \boldsymbol{A}$, the state-transition matrix Peano-Baker series becomes

$$\begin{aligned}
\boldsymbol{\Phi}(t,s) &= \sum_{k=0}^{\infty} \boldsymbol{M}_k(t,s) \\
&= \sum_{k=0}^{\infty} \int_s^t \int_s^{\tau_1} \cdots \int_s^{\tau_{k-1}} \underbrace{\boldsymbol{A}\boldsymbol{A}\cdots\boldsymbol{A}}_{k-\text{fold product}} \, d\tau_k \cdots d\tau_1 \\
&= \sum_{k=0}^{\infty} \boldsymbol{A}^k \int_s^t \int_s^{\tau_1} \cdots \int_s^{\tau_{k-1}} d\tau_k \cdots d\tau_1
\end{aligned}$$

To compute $\boldsymbol{M}_k(t,s)$, let's look at $k = 0, 1, 2, \ldots$ to see the pattern:

- What is $\boldsymbol{M}_0(t,s)$?
- What is $\boldsymbol{M}_1(t,s)$?
- What is $\boldsymbol{M}_2(t,s)$?
- What is $\boldsymbol{M}_3(t,s)$?

Important Special Case: $\boldsymbol{A}(t) \equiv \boldsymbol{A}$

By induction, we can show that

$$\boldsymbol{M}_k(t,s) \;=\; \boldsymbol{A}^k \frac{1}{k!}(t-s)^k$$

hence

$$\boldsymbol{\Phi}(t,s) = \sum_{k=0}^{\infty} \boldsymbol{A}^k \frac{1}{k!}(t-s)^k$$

Suppose, for $x \in \mathbb{C}$, we have

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

Math trivia question: What is $f(x)$?

Matrix Exponential

Definition (Matrix Exponential)

Given $\boldsymbol{W} \in \mathbb{C}^{n \times n}$, the matrix exponential is defined as

$$\exp(\boldsymbol{W}) = \sum_{k=0}^{\infty} \frac{\boldsymbol{W}^k}{k!}$$

Note that the matrix exponential is not performed element-by-element, i.e.

$$\exp\left(\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}\right) \neq \begin{bmatrix} e^{w_{11}} & e^{w_{12}} \\ e^{w_{21}} & e^{w_{22}} \end{bmatrix}$$

Matlab has a special function (`expm`) that computes matrix exponentials. Calling `exp(W)` will not give the same results as `expm(W)`.

Important Special Case: $A(t) \equiv A$

Putting it all together, when $\boldsymbol{A}(t) \equiv \boldsymbol{A}$, we can say that

$$\boldsymbol{\Phi}(t,s) = \exp\{(t-s)\boldsymbol{A}\}$$

Then the solution to the state-update DE is

$$\boldsymbol{x}(t) = \exp\{(t-t_0)\boldsymbol{A}\}\,\boldsymbol{x}(t_0) + \int_{t_0}^{t} \exp\{(t-\tau)\boldsymbol{A}\}\,\boldsymbol{B}(\tau)\boldsymbol{u}(\tau)\,d\tau$$

and the output equation is

$$\boldsymbol{y}(t) = \boldsymbol{C}(t)\exp\{(t-t_0)\boldsymbol{A}\}\,\boldsymbol{x}(t_0) + \boldsymbol{C}(t)\int_{t_0}^{t} \exp\{(t-\tau)\boldsymbol{A}\}\,\boldsymbol{B}(\tau)\boldsymbol{u}(\tau)\,d\tau + \boldsymbol{D}(t)\boldsymbol{u}(t)$$

Contrast/Comparison Between CT and DT Solutions

Similarities

- Results have same "look".
- Both have state transition matrices with same intuitive properties, e.g. semigroup.

Differences

- In DT systems, $\boldsymbol{x}[k]$ is only defined for $k \geq k_0$ because the DT-STM $\boldsymbol{\Phi}[k, k_0]$ is only defined for $k \geq k_0$.
- In CT systems, $\boldsymbol{x}(t)$ is only defined for all $t \in \mathbb{R}$ because the CT-STM $\boldsymbol{\Phi}(t, t_0)$ is defined for all $(t, t_0) \in \mathbb{R}^2$.
- We didn't prove this, but the CT-STM $\boldsymbol{\Phi}(t, t_0)$ is always invertible. This is not true of the DT-STM $\boldsymbol{\Phi}[k, k_0]$.

Conclusions

- Solution to LTI or LTV discrete-time state-space difference equations (existence and uniqueness)
- Solution to LTI or LTV continuous-time state-space differential equations (existence and uniqueness)
- Special case: time-invariant $\boldsymbol{A}$ matrix
- LTI discrete-time systems: $\boldsymbol{A}^{k-j}$
- LTI continuous-time systems: $\exp\{(t-\tau)\boldsymbol{A}\}$