ECE504: Lecture 5

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Lecture 5 Major Topics

We are still in Part II of ECE504: **Quantitative and qualitative analysis of systems**

mathematical description → results about behavior of system

Today:

1. Existence and uniqueness of solutions to state equations for continuous-time systems
2. Linear algebraic tools that we are going to need for analysis of $\boldsymbol{A}^k$ and $\exp\{\boldsymbol{A}t\}$.
    - Subspaces
    - Nullspace and range
    - Rank
    - Matrix invertibility

You should be reading Chen Chapter 4 now (and referring back to Chapter 3 for the necessary linear algebra).

Continuous-Time Linear Systems

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t) \quad (1)\\ \boldsymbol{y}(t) &= \boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{D}(t)\boldsymbol{u}(t) \quad (2)\end{aligned}$$

Theorem

For any $t_0 \in \mathbb{R}$, any $\boldsymbol{x}(t_0) \in \mathbb{R}^n$, and any $\boldsymbol{u}(t) \in \mathbb{R}^p$ for all $t \geq t_0$, there exists a unique solution $\boldsymbol{x}(t)$ for all $t \in \mathbb{R}$ to the state-update differential equation (1). It is given as

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{\Phi}(t, \tau)\boldsymbol{B}(\tau)\boldsymbol{u}(\tau)\, d\tau \quad t \in \mathbb{R}$$

where $\boldsymbol{\Phi}(t, s) : \mathbb{R}^2 \mapsto \mathbb{R}^{n \times n}$ is the unique function satisfying

$$\frac{d}{dt}\boldsymbol{\Phi}(t, s) = \boldsymbol{A}(t)\boldsymbol{\Phi}(t, s) \text{ with } \boldsymbol{\Phi}(s, s) = \boldsymbol{I}_n.$$

Theorem Remarks

- Note that this theorem claims two things:
  1. A solution to the state-update equation always **exists**.
  2. The solution is **unique**.
- Why is this important?
  - Not every differential equation has a solution, e.g.

$$\dot{x}(t) = \frac{1}{t} \text{ with } x(0) = 5$$

  - Not every differential equation has a unique solution

$$\dot{x}(t) = 3(x(t))^{2/3} \text{ with } x(0) = 0$$

- We have already established **uniqueness** for the vector/matrix state update differential equation $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t)$ with initial condition $\boldsymbol{x}(t_0)$.
- We still need to establish **existence**.

Theorem: Existence Proof Warmup #1

To develop some intuition, let's first assume that everything is scalar, i.e. $p = q = n = 1$. Our state update equation becomes

$$\dot{x}(t) = a(t)x(t) + b(t)u(t)$$

Let

$$\phi(t,s) := \exp\left\{\int_s^t a(\tau)\,d\tau\right\}$$

What is $\phi(s,s)$?

What is $\frac{d}{dt}\phi(t,s)$?

Theorem: Existence Proof Warmup #1

Note that $\phi(t,s) = \exp\left\{\int_s^t a(\tau)\,d\tau\right\}$ always exists and satisfies its own differential equation:

$$\frac{d}{dt}\phi(t,s) = a(t)\phi(t,s) \text{ with } \phi(s,s) = 1.$$

Now lets try the following solution to the scalar state-update differential equation with initial state condition $x(t_0)$:

$$x(t) \;=\; \phi(t,t_0)x(t_0) + \int_{t_0}^t \phi(t,\tau)b(\tau)u(\tau)\,d\tau \qquad \forall t \in \mathbb{R}$$

To see that this is indeed a solution, we need to confirm two things:

1. Does our solution satisfy the initial condition requirement of the scalar state-update DE?
2. Does our solution really solve the scalar state-update DE?

Theorem: Existence Proof Warmup #2

To develop additional intuition, let's now assume that everything is time-invariant, i.e. $\boldsymbol{A}(t) \equiv \boldsymbol{A}$ and $\boldsymbol{B}(t) \equiv \boldsymbol{B}$. Our state update equation becomes

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$

Let

$$\boldsymbol{\Phi}(t,s) := \sum_{k=0}^{\infty} \boldsymbol{A}^k \frac{1}{k!}(t-s)^k$$

What is $\boldsymbol{\Phi}(s,s)$?

What is $\frac{d}{dt}\boldsymbol{\Phi}(t,s)$?

Theorem: Existence Proof Warmup #2

Note that $\boldsymbol{\Phi}(t,s) = \sum_{k=0}^{\infty} \boldsymbol{A}^k \frac{1}{k!}(t-s)^k$ exists for any $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, $t \in \mathbb{R}$, and $s \in \mathbb{R}$. Moreover, $\boldsymbol{\Phi}(t,s)$ satisfies its own differential equation:

$$\frac{d}{dt}\boldsymbol{\Phi}(t,s) = \boldsymbol{A}\boldsymbol{\Phi}(t,s) \text{ with } \boldsymbol{\Phi}(s,s) = \boldsymbol{I}_n.$$

Now lets try the following solution to the scalar state-update differential equation with initial state condition $\boldsymbol{x}(t_0)$:

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t,t_0)\boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{\Phi}(t,\tau)\boldsymbol{B}(\tau)\boldsymbol{u}(\tau)\,d\tau \qquad \forall t \in \mathbb{R}$$

To see that this is indeed a solution, we need to confirm two things:

1. Does our solution satisfy the initial condition requirement of the scalar state-update DE?
2. Does our solution really solve the scalar state-update DE?

Theorem: Existence Proof for General Case

For the general (non-scalar, time-varying) case, we propose the solution

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{\Phi}(t, \tau)\boldsymbol{B}(\tau)\boldsymbol{u}(\tau)\, d\tau \qquad (3)$$

where the state transition matrix satisfies the matrix differential equation

$$\frac{d}{dt}\boldsymbol{\Phi}(t, s) = \boldsymbol{A}(t)\boldsymbol{\Phi}(t, s) \text{ with } \boldsymbol{\Phi}(s, s) = \boldsymbol{I}_n. \qquad (4)$$

Note that (4) is consistent with our two warmup cases.

To complete the existence proof, we need to:

1. Show that (3) with $\boldsymbol{\Phi}(t, s)$ defined according to (4) satisfies the initial condition requirement of the state-update DE.
2. Show that (3) with $\boldsymbol{\Phi}(t, s)$ defined according to (4) is indeed a solution to the state-update DE.
3. Show that there always exists a solution to the matrix DE (4).

Theorem: Existence Proof for General Case: Part 1

Show that (3) with $\mathbf{\Phi}(t, s)$ defined according to (4) satisfies the initial condition requirement of the state-update DE.

Theorem: Existence Proof for General Case: Part 2

Show that (3) with $\mathbf{\Phi}(t,s)$ defined according to (4) is indeed a solution to the state-update DE.

Theorem: Existence Proof for General Case: Part 3

Show that there always exists a solution to the matrix DE (4).

Peano-Baker Series Example

$$\boldsymbol{A}(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix}$$

Fundamental Matrix Method

While the Peano-Baker series establishes existence (and thus concludes the proof of the existence and uniqueness theorem), it is sometimes easier to find $\mathbf{\Phi}(t, s)$ via the "fundamental matrix method" (Chen section 4.5).

Basic idea:

1. Consider the the continuous time DE with $\boldsymbol{x}(t) \in \mathbb{R}^n$

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) \tag{5}$$

2. Choose n different initial conditions $\boldsymbol{x}_1(t_0), \dots, \boldsymbol{x}_n(t_0)$. These n initial condition vectors must be linearly independent.
3. These n different initial conditions lead to n different solutions to (5). Call these solutions $\boldsymbol{x}_1(t), \dots, \boldsymbol{x}_n(t)$ and put them into a matrix $\boldsymbol{X}(t) = [\boldsymbol{x}_1(t), \dots, \boldsymbol{x}_n(t)] \in \mathbb{R}^{n \times n}$.
4. Note that $\dot{\boldsymbol{X}}(t) = \boldsymbol{A}(t)\boldsymbol{X}(t)$. The quantity $\boldsymbol{X}(t)$ is called a fundamental matrix of (5). Is the fundamental matrix unique?

Fundamental Matrix Method

Let $\boldsymbol{X}(t)$ be any fundamental matrix of (5). Note that $\boldsymbol{X}(t)$ is invertible for all t (see Chen p. 107). The state transition matrix $\boldsymbol{\Phi}(t,s)$ can then be computed as

$$\boldsymbol{\Phi}(t,s) = \boldsymbol{X}(t)\boldsymbol{X}^{-1}(s).$$

Check:

$$\boldsymbol{\Phi}(s,s) =$$

$$\frac{d}{dt}\boldsymbol{\Phi}(t,s) =$$

Fundamental Matrix Example

$$\boldsymbol{A}(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix}$$

Remarks on the CT State-Transition Matrix $\mathbf{\Phi}(t,s)$

1. There are many ways to compute $\mathbf{\Phi}(t,s)$. Some are easier than others, but computing $\mathbf{\Phi}(t,s)$ is almost always difficult.
2. Do different methods for computing $\mathbf{\Phi}(t,s)$ lead to different solutions?
3. Unlike the DT-STM $\mathbf{\Phi}[k,j]$, the CT-STM $\mathbf{\Phi}(t,s)$ is defined for any $(t,s) \in \mathbb{R}^2$. This means that we can specify an initial state $\boldsymbol{x}(t_0)$ and compute the system response at times **prior** to t_0.
4. It is easy to show that $\mathbf{\Phi}(t,s)$ possesses the semi-group property, i.e.

$$\mathbf{\Phi}(t,\tau) = \mathbf{\Phi}(t,s)\mathbf{\Phi}(s,\tau)$$

for any $(t,\tau,s) \in \mathbb{R}^3$ from the fundamental matrix formulation:

$$\mathbf{\Phi}(t,\tau) = \mathbf{\Phi}(t,s)\mathbf{\Phi}(s,\tau) = \boldsymbol{X}(t)\boldsymbol{X}^{-1}(s)\boldsymbol{X}(s)\boldsymbol{X}^{-1}(\tau) = \boldsymbol{X}(t)\boldsymbol{X}^{-1}(\tau)$$

Important Special Case: $A(t) \equiv A$

When $\boldsymbol{A}(t) \equiv \boldsymbol{A}$, the state-transition matrix Peano-Baker series becomes

$$\begin{aligned}
\boldsymbol{\Phi}(t,s) &= \sum_{k=0}^{\infty} \boldsymbol{M}_k(t,s) \\
&= \sum_{k=0}^{\infty} \int_s^t \int_s^{\tau_1} \cdots \int_s^{\tau_{k-1}} \underbrace{\boldsymbol{A}\boldsymbol{A}\cdots\boldsymbol{A}}_{k-\text{fold product}} \; d\tau_k \cdots d\tau_1 \\
&= \sum_{k=0}^{\infty} \boldsymbol{A}^k \int_s^t \int_s^{\tau_1} \cdots \int_s^{\tau_{k-1}} d\tau_k \cdots d\tau_1
\end{aligned}$$

To compute $\boldsymbol{M}_k(t,s)$, let's look at $k = 0, 1, 2, \dots$ to see the pattern...

Important Special Case: $A(t) \equiv A$

By induction, we can show that

$$M_k(t,s) \quad = \quad A^k \frac{1}{k!}(t-s)^k$$

hence

$$\Phi(t,s) = \sum_{k=0}^{\infty} A^k \frac{1}{k!}(t-s)^k$$

which is consistent with our earlier result (warmup #2).

Suppose, for $x \in \mathbb{C}$, we have

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

What is $f(x)$?

Matrix Exponential

Definition (Matrix Exponential)

Given $\boldsymbol{W} \in \mathbb{C}^{n \times n}$, the matrix exponential is defined as

$$\exp(\boldsymbol{W}) \;=\; \sum_{k=0}^{\infty} \frac{\boldsymbol{W}^k}{k!}$$

Note that the matrix exponential is not performed element-by-element, i.e.

$$\exp\left(\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}\right) \neq \begin{bmatrix} e^{w_{11}} & e^{w_{12}} \\ e^{w_{21}} & e^{w_{22}} \end{bmatrix}$$

Matlab has a special function (`expm`) that computes matrix exponentials. Calling `exp(W)` will not give the same results as `expm(W)`.

Important Special Case: $A(t) \equiv A$

Putting it all together, when $\boldsymbol{A}(t) \equiv \boldsymbol{A}$, we can say that

$$\boldsymbol{\Phi}(t,s) = \sum_{k=0}^{\infty} \boldsymbol{A}^k \frac{1}{k!}(t-s)^k = \exp\{(t-s)\boldsymbol{A}\}$$

Then the solution to the LTI continuous-time state-update DE is

$$\boxed{\boldsymbol{x}(t) = \exp\{(t-t_0)\boldsymbol{A}\}\,\boldsymbol{x}(t_0) + \int_{t_0}^{t} \exp\{(t-\tau)\boldsymbol{A}\}\,\boldsymbol{B}(\tau)\boldsymbol{u}(\tau)\,d\tau}$$

and the output equation is

$$\boxed{\boldsymbol{y}(t) = \boldsymbol{C}(t)\exp\{(t-t_0)\boldsymbol{A}\}\,\boldsymbol{x}(t_0) + \boldsymbol{C}(t)\int_{t_0}^{t} \exp\{(t-\tau)\boldsymbol{A}\}\,\boldsymbol{B}(\tau)\boldsymbol{u}(\tau)\,d\tau + \boldsymbol{D}(t)\boldsymbol{u}(t)}$$

Contrast/Comparison Between CT and DT Solutions

Similarities

- CT and DT solutions have same "look".
- CT and DT solutions have state transition matrices with same intuitive properties, e.g. semigroup.

Differences

- In DT systems, $\boldsymbol{x}[k]$ is only defined for $k \geq k_0$ because the DT-STM $\boldsymbol{\Phi}[k, k_0]$ is only defined for $k \geq k_0$.
- In CT systems, $\boldsymbol{x}(t)$ is only defined for all $t \in \mathbb{R}$ because the CT-STM $\boldsymbol{\Phi}(t, t_0)$ is defined for all $(t, t_0) \in \mathbb{R}^2$.
- We didn't prove this, but the CT-STM $\boldsymbol{\Phi}(t, t_0)$ is always invertible. This is not true of the DT-STM $\boldsymbol{\Phi}[k, k_0]$.

What We Know

- We know how to **solve** discrete-time LTV and LTI systems. "Solve" means "write an analytical expression for $\boldsymbol{x}[k]$ and $\boldsymbol{y}[k]$ given $\boldsymbol{A}[k]$, $\boldsymbol{B}[k]$, $\boldsymbol{C}[k]$, $\boldsymbol{D}[k]$, and $\boldsymbol{x}[k_0]$".
- We know that solutions must exist and must be unique.

- We know how to **solve** continuous-time LTV and LTI systems. "Solve" means "write an analytical expression for $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ given $\boldsymbol{A}(t)$, $\boldsymbol{B}(t)$, $\boldsymbol{C}(t)$, $\boldsymbol{D}(t)$, and $\boldsymbol{x}(t_0)$".
- We know that solutions must exist and must be unique.
- We also know two ways to compute the state transition matrix.

- We know some of the properties of state transition matrices.
- We know differences between the DT-STM and the CT-STM.

Where We Are Heading

Our focus is going to shift primarily to LTI systems for a little while.

Recall that, when $\boldsymbol{A}$ is not a function of time, the state transition matrices become

$$\begin{aligned} \boldsymbol{\Phi}[k,j] &= \boldsymbol{A}^{k-j} \text{ (discrete time)} \\ \boldsymbol{\Phi}(t,s) &= \exp\{(t-s)\boldsymbol{A}\} \text{ (continuous time)} \end{aligned}$$

We would like to be able to better analyze these matrix functions in order to, for example, efficiently compute $\boldsymbol{A}^{k-j}$.

We are going to need to learn some more linear algebra first...

Sets and Subspaces

Let $\mathcal{A}$ and $\mathcal{B}$ be sets.

- $\mathcal{A} \subset \mathcal{B}$ means that all elements of the set $\mathcal{A}$ are also in the set $\mathcal{B}$.
- $x \in \mathcal{A}$ to mean that x is an element of the set $\mathcal{A}$.
- $\mathcal{A} \subset \mathcal{B}$ and $x \in \mathcal{A}$ implies that $x \in \mathcal{B}$.

Definition

$\mathcal{S} \subset \mathbb{R}^n$ is a subspace if and only if $\mathcal{S}$ is closed under addition and scalar multiplication, i.e.

$$\begin{array}{ccc} \boldsymbol{x} \in \mathcal{S} \text{ and } \boldsymbol{y} \in \mathcal{S} & \Rightarrow & \boldsymbol{x} + \boldsymbol{y} \in \mathcal{S} \\ & \text{and} & \\ \boldsymbol{x} \in \mathcal{S} \text{ and } \alpha \in \mathbb{R} & \Rightarrow & \alpha \boldsymbol{x} \in \mathcal{S}. \end{array}$$

Note that subspaces must always include the zero vector.

Spanning Set of a Subspace

Definition

A spanning set for the subspace $\mathcal{S} \subset \mathbb{R}^n$ is a set of vectors $\boldsymbol{s}_1, \ldots \boldsymbol{s}_p$, each in $\mathcal{S}$, such that every element of $\mathcal{S}$ can be expressed as a linear combination of the vectors $\boldsymbol{s}_1, \ldots \boldsymbol{s}_p$, i.e.

$$\boldsymbol{x} \in \mathcal{S} \quad \Rightarrow \quad \text{there exists } \alpha_1, \ldots, \alpha_p \text{ such that } \boldsymbol{x} = \alpha_1 \boldsymbol{s}_1 + \cdots + \alpha_p \boldsymbol{s}_p$$

where $\alpha_i \in \mathbb{R}$ for $i = 1, \ldots, p$.

Example: Suppose $\mathcal{S}$ is the xy plane in $\mathbb{R}^3$. Which of the following are spanning sets?

$$\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\} \text{ or } \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$$

Some Facts About Subspaces, Spanning Sets, and Bases

1. Every $\mathcal{S} \subset \mathbb{R}^n$ possesses a linearly independent spanning set. Such a set is called a **basis** for $\mathcal{S}$. This basis is not unique, of course.
2. The number of vectors in any basis for $\mathcal{S}$ is the same. This number is called the **dimension** of $\mathcal{S}$. We use the notation $\dim(\mathcal{S})$ to denote the dimension of a subspace.
3. If $\mathcal{S}$ is a subspace of $\mathbb{R}^n$, then $\dim(\mathcal{S}) \leq n$ with equality if and only if $\mathcal{S} = \mathbb{R}^n$.
4. Any spanning set for $\mathcal{S}$ contains at least $\dim(\mathcal{S})$ vectors.
5. Any set with elements from $\mathcal{S}$ containing more than $\dim(\mathcal{S})$ vectors is linearly dependent.
6. A basis is a minimally-sized spanning set of $\mathcal{S}$.
7. A basis is a maximally-sized linear independent set of vectors in $\mathcal{S}$.

Nullspace and Range

Given $\boldsymbol{W} \in \mathbb{R}^{m \times n}$ (not necessarily square), there are two important subspaces related to this matrix.

Definition

The **nullspace** of $\boldsymbol{W}$ is defined as the set of all $\boldsymbol{x} \in \mathbb{R}^n$ such that $\boldsymbol{W}\boldsymbol{x} = \boldsymbol{0}$. We denote this subspace of $\mathbb{R}^n$ as $\mathrm{null}(\boldsymbol{W})$.

Definition

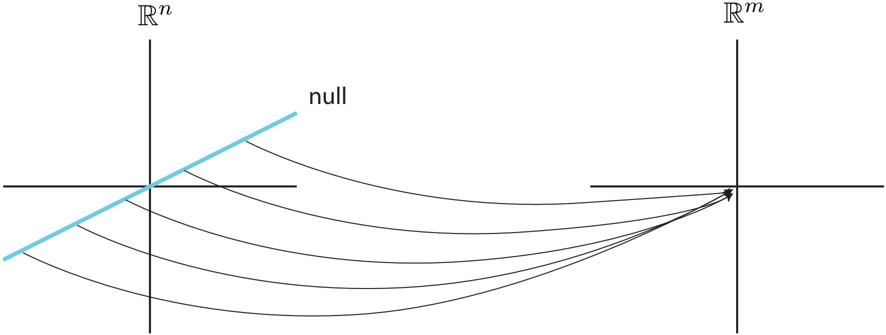
The **range** of $\boldsymbol{W}$ is defined as the set of all $\boldsymbol{y} \in \mathbb{R}^m$ such that there exists an $\boldsymbol{x}$ satisfying $\boldsymbol{W}\boldsymbol{x} = \boldsymbol{y}$. We denote this subspace of $\mathbb{R}^m$ as $\mathrm{range}(\boldsymbol{W})$.

The range is also sometimes called the "column space" because it is the subspace generated by linear combinations of the columns of $\boldsymbol{W}$.

Note that both subspaces always include the zero vector.

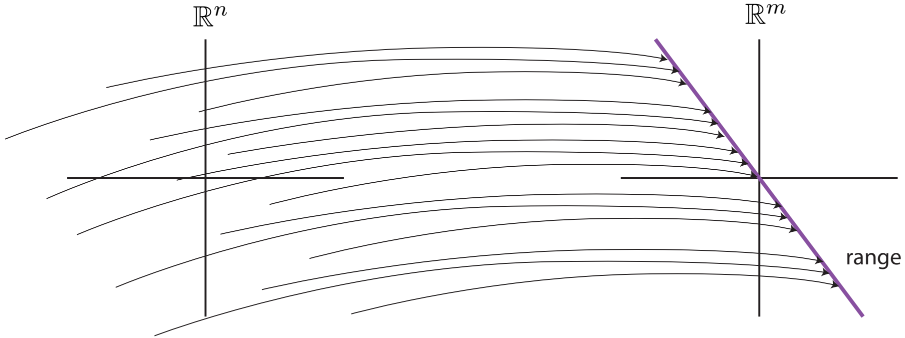
Nullspace

The matrix $\boldsymbol{W} \in \mathbb{R}^{m \times n}$ maps vectors from $\mathbb{R}^n$ to $\mathbb{R}^m$. The nullspace of $\boldsymbol{W}$ is a subspace of $\mathbb{R}^n$.



Range

The matrix $\boldsymbol{W} \in \mathbb{R}^{m \times n}$ maps vectors from $\mathbb{R}^n$ to $\mathbb{R}^m$. The range of $\boldsymbol{W}$ is a subspace of $\mathbb{R}^m$.



Nullspace and Range Examples

Suppose

$$\boldsymbol{W} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \tag{6}$$

$$\boldsymbol{W} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \tag{7}$$

$$\boldsymbol{W} = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \tag{8}$$

What is the nullspace and range of $\boldsymbol{W}$ in each case?

Existence and Uniqueness of Solutions to $\boldsymbol{Ax} = \boldsymbol{b}$

For any $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$, a solution to $\boldsymbol{Ax} = \boldsymbol{b}$ exists if and only if $\boldsymbol{b} \in \mathrm{range}(\boldsymbol{A})$.

For any $\boldsymbol{A} \in \mathbb{R}^{m \times n}$ and $\boldsymbol{b} \in \mathbb{R}^m$, a solution to $\boldsymbol{Ax} = \boldsymbol{b}$ is unique if and only if $\dim(\mathrm{null}(\boldsymbol{A})) = 0$.

Gaussian Elimination and Echelon Form

- GE is an algorithm for reducing a matrix to **echelon form**.
- Once you have a matrix in echelon form, you can easily determine its range and the dimension of its nullspace.
- This allows you to easily answer questions about the existence and uniqueness of solutions to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$.

1. Form "augmented matrix" $\boldsymbol{U} = [\boldsymbol{A} \mid \boldsymbol{b}] \in \mathbb{R}^{m \times n+1}$.
2. Notation $\boldsymbol{U}(k,:)$ is the k^{th} row of $\boldsymbol{U}$ and $\boldsymbol{U}(k,j)$ is the k, j^{th} element of $\boldsymbol{U}$.
3. Force $\boldsymbol{U}(2,1) = 0$ by forming an appropriate combination of other rows and subtracting this combination from $\boldsymbol{U}(2,:)$.
4. Force $\boldsymbol{U}(3,1) = \boldsymbol{U}(3,2) = 0$ using the same technique.
5. Keep doing this until you have an upper triangular matrix.
6. You can now solve the last row since it has only one unknown.
7. Back substitute your answer and solve the second last row.
8. Keep doing this until you solve the top row.

Gaussian Elimination and Echelon Form Examples

Using the Echelon Form to Determine a Basis for $\mathrm{range}(\boldsymbol{A})$

The pivot columns of $\boldsymbol{A}$ for a basis for the range of $\boldsymbol{A}$. Note that the echelon form matrix tells you which columns to pick from $\boldsymbol{A}$. Example:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & -1 & 3 & 0 \\ -1 & -2 & 2 & -2 & -1 \\ 1 & 2 & 0 & 4 & 0 \\ 0 & 0 & 2 & 2 & -1 \end{bmatrix} \text{ and } \boldsymbol{b} = \begin{bmatrix} 1 \\ 1 \\ 6 \\ 7 \end{bmatrix} \tag{9}$$

After reduction to echelon form of $\boldsymbol{U} = [\boldsymbol{A} \mid \boldsymbol{b}]$, we have

$$\begin{bmatrix} 1 & 2 & -1 & 3 & 0 & 1 \\ 0 & 0 & 1 & 1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{10}$$

The pivot columns here are the first, third, and fifth. Continued...

Using the Echelon Form to Determine a Basis for $\mathrm{range}(\boldsymbol{A})$

Hence a basis for the range of $\boldsymbol{A}$ is

$$\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

You should be able to verify that these vectors are linearly independent.

The range of $\boldsymbol{A}$ is the subspace formed by all linear combinations of these vectors. Solutions to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ exist only when $\boldsymbol{b} \in \mathrm{range}(\boldsymbol{A})$.

Using the Echelon Form to Determine $\dim(\text{null}(\boldsymbol{A}))$

The dimension of the nullspace of $\boldsymbol{A}$ (also called the **nullity**) is simply the number of non-pivot columns in the echelon form.

You can use the echelon form to also find a basis for $\text{null}(\boldsymbol{A})$ (see any good linear algebra textbook for the details). In our example, a basis for the nullspace is

$$\left\{ \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -4 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

You should be able to verify that these vectors are linearly independent and that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$ if $\boldsymbol{x}$ is any linear combination of these basis vectors.

Most importantly, solutions to $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$ are unique only when $\dim(\text{null}(\boldsymbol{A})) = 0$.

Rank

Definition

The rank of $\boldsymbol{W}$ is defined as the dimension of the range of $\boldsymbol{W}$, i.e.

$$\text{rank}(\boldsymbol{W}) \quad := \quad \dim(\text{range}(\boldsymbol{W})).$$

Some useful facts:

- For $\boldsymbol{W} \in \mathbb{R}^{m \times n}$, $0 \leq \text{rank}(\boldsymbol{W}) \leq \min\{m, n\}$.
- $\text{rank}(\boldsymbol{W})$ is equal to the number of pivot columns in the echelon form of $\boldsymbol{W}$.
- Since $\dim(\text{null}(\boldsymbol{W}))$ is equal to the number of non-pivot columns in the echelon form, rank + nullity must equal n.
- $0 \leq \text{rank}(\boldsymbol{U}\boldsymbol{W}) \leq \min\{\text{rank}(\boldsymbol{U}), \text{rank}(\boldsymbol{W})\}$. In other words, matrix multiplication can only decrease rank.

Matrix Transpose

Definition

Given

$$\boldsymbol{W} = \begin{bmatrix} w_{11} & w_{12} & \dots & w_{1n} \\ w_{21} & w_{22} & \dots & w_{2n} \\ \vdots & \vdots & & \vdots \\ w_{m1} & w_{m2} & \dots & w_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

the transpose of $\boldsymbol{W}$ is given as

$$\boldsymbol{W}^\top = \begin{bmatrix} w_{11} & w_{21} & \dots & w_{m1} \\ w_{12} & w_{22} & \dots & w_{m2} \\ \vdots & \vdots & & \vdots \\ w_{1n} & w_{2n} & \dots & w_{mn} \end{bmatrix} \in \mathbb{R}^{n \times m}.$$

An Important Property of the Matrix Transpose

For any $\boldsymbol{A} \in \mathbb{R}^{m \times p}$ and $\boldsymbol{B} \in \mathbb{R}^{p \times n}$, the product $\boldsymbol{C} = \boldsymbol{A}\boldsymbol{B}$ is an $m \times n$ real-valued matrix. The transpose of $\boldsymbol{C}$ is

$$\boldsymbol{C}^\top = (\boldsymbol{A}\boldsymbol{B})^\top = \boldsymbol{B}^\top \boldsymbol{A}^\top \in \mathbb{R}^{n \times m}$$

Note that the order of the matrix product has been changed by the transpose. Do the matrix dimensions agree?

Invertibility of Square Matrices

Definition

Given $\boldsymbol{W} \in \mathbb{R}^{n \times n}$, we say that $\boldsymbol{W}$ is invertible if there exists $\boldsymbol{V} \in \mathbb{R}^{n \times n}$ such that $\boldsymbol{V}\boldsymbol{W} = \boldsymbol{W}\boldsymbol{V} = \boldsymbol{I}_n$. The quantity $\boldsymbol{V}$ is called the matrix inverse for $\boldsymbol{W}$ and we use the notation: $\boldsymbol{V} = \boldsymbol{W}^{-1}$.

The matrix inverse does not always exist, but when it does, it is unique.

Fact: If $\boldsymbol{W}$ is invertible, then $\boldsymbol{W}^\top$ is also invertible. To see this, just use what you know about the matrix inverse and the matrix transpose

$$\begin{aligned} \boldsymbol{W}\boldsymbol{W}^{-1} &= \boldsymbol{I}_n \\ (\boldsymbol{W}\boldsymbol{W}^{-1})^\top &= \boldsymbol{I}_n^\top \\ (\boldsymbol{W}^{-1})^\top \boldsymbol{W}^\top &= \boldsymbol{I}_n \end{aligned}$$

and, by the definition, $(\boldsymbol{W}^{-1})^\top = (\boldsymbol{W}^\top)^{-1}$.

Invertibility of Square Matrices: Equivalences

The following statements are equivalent:

1. $\boldsymbol{W}$ is invertible.
2. The only $\boldsymbol{x} \in \mathbb{R}^n$ satisfying $\boldsymbol{W}\boldsymbol{x} = \mathbf{0}$ is $\boldsymbol{x} = 0$.
3. For every $\boldsymbol{b} \in \mathbb{R}^n$, there exists a unique $\boldsymbol{x} \in \mathbb{R}^n$ solving $\boldsymbol{W}\boldsymbol{x} = \boldsymbol{b}$.
4. The echelon form of $\boldsymbol{W}$ has no rows composed of all zeros.
5. $\det(\boldsymbol{W}) \neq 0$.
6. $\text{rank}(\boldsymbol{W}) = n$.

Proofs...

Conclusions

1. Existence and uniqueness of solutions to CT and DT systems.
2. Proofs were constructive: you can now "solve" these systems.
3. Linear algebra tools to lay foundation for analysis of $\boldsymbol{A}^k$ and $\exp(\boldsymbol{A})$:
    - Subspaces
    - Nullspace and range
    - Rank
    - Matrix invertibility equivalences