ECE504: Lecture 6

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Lecture 6 Major Topics

We are still in Part II of ECE504: **Quantitative and qualitative analysis of systems**

mathematical description → results about behavior of system

Our focus today is on linear time-invariant systems. In this case, recall that the DT-STM is $\boldsymbol{\Phi}[k,j] = \boldsymbol{A}^{k-j}$ and the CT-STM is $\boldsymbol{\Phi}(t,s) = \exp\{(t-s)\boldsymbol{A}\}$. We will discuss

1. Properties of $\boldsymbol{A}^k$ and $\exp\{t\boldsymbol{A}\}$.
2. Eigenvalues and eigenvectors.
3. Computation of $\boldsymbol{A}^k$ and $\exp\{t\boldsymbol{A}\}$ when $\boldsymbol{A}$ is diagonalizable.
4. Computation of $\boldsymbol{A}^k$ and $\exp\{t\boldsymbol{A}\}$ when $\boldsymbol{A}$ is not diagonalizable.

You should be finishing Chen Chapter 4 now (and referring back to Chapter 3 as necessary). You should also look over "Nineteen Dubious Ways to Compute the Exponential of a Matrix, Twenty-Five Years Later" by Moler and Van Loan (link on course web page).

Basic Properties of $\boldsymbol{A}^k$ and $\exp\{t\boldsymbol{A}\}$

Recall that the matrix exponential $\exp\{t\boldsymbol{A}\}$ is not performed element-by-element. Nevertheless, the matrix exponential has many of the same properties as the usual scalar exponential. Specifically:

1. For any $\boldsymbol{A} \in \mathbb{R}^{n \times n}$

$$\lim_{t \to 0} \exp\{t\boldsymbol{A}\} = \boldsymbol{I}_n$$

   This can be seen directly from the definition of $\exp\{t\boldsymbol{A}\}$.

2. For any $\boldsymbol{A} \in \mathbb{R}^{n \times n}$

$$\exp\{(t_1 + t_2)\boldsymbol{A}\} = \exp\{t_1\boldsymbol{A}\}\exp\{t_2\boldsymbol{A}\}$$

   This is a consequence of the semigroup property of $\boldsymbol{\Phi}(t, s)$.

3. Given $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $\tilde{\boldsymbol{A}} \in \mathbb{R}^{n \times n}$, does

$$\exp\{t(\boldsymbol{A} + \tilde{\boldsymbol{A}})\} = \exp\{t\boldsymbol{A}\}\exp\{t\tilde{\boldsymbol{A}}\}?$$

Basic Properties of A^k and $\exp\{tA\}$ (cont.)

4. Given any $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ such that

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & a_{nn} \end{bmatrix} \text{ is diagonal, then } \exp\{t\boldsymbol{A}\} = \begin{bmatrix} e^{a_{11}t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{a_{nn}t} \end{bmatrix}$$

   This can be seen directly from the definition of $\exp\{t\boldsymbol{A}\}$.

5. Given any $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ such that we can find some invertible $\boldsymbol{V} \in \mathbb{C}^{n\times n}$ satisfying

$$\boldsymbol{V}^{-1}\boldsymbol{A}\boldsymbol{V} = \boldsymbol{\Lambda} = \begin{bmatrix} \lambda_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_{nn} \end{bmatrix}$$

   then

$$\boldsymbol{A}^k = \underbrace{(\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1})\dots(\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1})}_{k-\text{fold product}} = \boldsymbol{V}\boldsymbol{\Lambda}^k\boldsymbol{V}^{-1}$$

Basic Properties of A^k and $\exp\{tA\}$ (cont.)

6. Given any $\boldsymbol{A} \in \mathbb{R}^{n\times n}$ such that we can find some invertible $\boldsymbol{V} \in \mathbb{C}^{n\times n}$ satisfying

$$\boldsymbol{V}^{-1}\boldsymbol{A}\boldsymbol{V} = \boldsymbol{\Lambda} = \begin{bmatrix} \lambda_{11} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_{nn} \end{bmatrix}$$

then

$$\begin{aligned} \exp\{t\boldsymbol{A}\} &= \sum_{k=0}^{\infty} \frac{t^k}{k!}(\boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1})^k = \boldsymbol{V}\left[\sum_{k=0}^{\infty} \frac{t^k}{k!}\boldsymbol{\Lambda}^k\right]\boldsymbol{V}^{-1} \\ &= \boldsymbol{V}\exp\{t\boldsymbol{\Lambda}\}\boldsymbol{V}^{-1} \\ &= \boldsymbol{V}\begin{bmatrix} e^{\lambda_{11}t} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda_{nn}t} \end{bmatrix}\boldsymbol{V}^{-1} \end{aligned}$$

Diagonalizability of Square Matrices

Diagonalizability makes the computation of $\boldsymbol{A}^k$ and $\exp\{t\boldsymbol{A}\}$ easy!

Question: Is every matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ diagonalizable?

Question: Is there any connection between invertibility and diagonalizability?

Question: What procedure can we use to diagonalize square matrices?

Eigenvalues

Definition

Given $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, λ_0 is an eigenvalue of $\boldsymbol{A}$ if and only if $(\boldsymbol{A} - \boldsymbol{I}_n \lambda_0)$ is not invertible.

Equivalently, based on what we already know about invertibility, we can say

$$\lambda_0 \text{ is an eigenvalue of } \boldsymbol{A} \Leftrightarrow \det(\boldsymbol{A} - \boldsymbol{I}_n \lambda_0) \neq 0$$

Definition

The characteristic polynomial of $\boldsymbol{A}$ is $\det(\lambda \boldsymbol{I}_n - \boldsymbol{A})$ where λ is a variable.

What is $\deg(\det(\lambda \boldsymbol{I}_n - \boldsymbol{A}))$?

It is not too hard to show that the eigenvalues of $\boldsymbol{A}$ are equivalent to the roots of the characteristic polynomial of $\boldsymbol{A}$.

Some Consequences of What We Know About Eigenvalues

1. There can be at most n different eigenvalues of $\boldsymbol{A}$.
2. Even if $\boldsymbol{A}$ is real, its eigenvalues can be complex.
3. If $\boldsymbol{A}$ is real and λ_0 is a complex eigenvalue of $\boldsymbol{A}$, then λ_0^* is also an eigenvalue of $\boldsymbol{A}$ where the notation $()^*$ means complex conjugate, i.e.

$$z = a + jb \quad \Leftrightarrow \quad z^* = a - jb$$

4. If λ_0 is an eigenvalue of $\boldsymbol{A}$, then $\dim(\text{null}(\boldsymbol{A} - \lambda_0 \boldsymbol{I}_n)) \geq 1$ since $\boldsymbol{A} - \lambda_0 \boldsymbol{I}_n$ is not invertible.

This last consequence is of particular importance. Let $\boldsymbol{v}_0$ be any vector (except for the zero vector) in the nullspace of $\boldsymbol{A} - \lambda_0 \boldsymbol{I}_n$. Then we can say that

$$(\boldsymbol{A} - \lambda_0 \boldsymbol{I}_n)\boldsymbol{v}_0 = \boldsymbol{0}$$

which can be rewritten as $\boldsymbol{A}\boldsymbol{v}_0 = \boldsymbol{\lambda}_0 \boldsymbol{v}_0$.

Eigenvectors

Definition

Given $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, $\boldsymbol{v}_0$ is an eigenvector of $\boldsymbol{A}$ corresponding to the eigenvalue λ_0 if and only if $\boldsymbol{A}\boldsymbol{v}_0 = \boldsymbol{\lambda}_0 \boldsymbol{v}_0$.

Intuition: The matrix $\boldsymbol{A}$ scales its eigenvectors by its eigenvalues.

Note that there is nothing unique about an eigenvector. If $\boldsymbol{v}_0$ is an eigenvector of $\boldsymbol{A}$ corresponding to the eigenvalue λ_0, then so is $\alpha \boldsymbol{v}_0$ for any $\alpha \neq 0$ since

$$\boldsymbol{A}(\alpha \boldsymbol{v}_0) = \alpha(\boldsymbol{A}\boldsymbol{v}_0) = \alpha(\boldsymbol{\lambda}_0 \boldsymbol{v}_0) = \boldsymbol{\lambda}_0(\alpha \boldsymbol{v}_0)$$

Linear Independence of Eigenvectors

Fact: If $\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_s \in \mathbb{R}^n$ are eigenvectors of $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ corresponding respectively with different eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_s \in \mathbb{C}$, then $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_s\}$ is a linearly independent set.

Intuitively: Eigenvectors corresponding to different eigenvalues are linearly independent.

Geometrically: Let $\mathcal{V}_j = \mathrm{null}(\boldsymbol{A} - \lambda_j \boldsymbol{I}_n)$. If $\lambda_j \neq \lambda_m$, then $\mathcal{V}_j \bigcap \mathcal{V}_m = \{\boldsymbol{0}\}$.

Important special case: If the matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ has n distinct eigenvalues, then there must exist n linearly independent eigenvectors $\{\boldsymbol{v}_1, \boldsymbol{v}_2, \dots, \boldsymbol{v}_n\} \in \mathbb{R}^n$. Let

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{v}_1 & \boldsymbol{v}_2 & \dots & \boldsymbol{v}_n \end{bmatrix}.$$

What can we say about the invertibility of $\boldsymbol{V}$?

When is A is Diagonalizable?

We now know that, when $\boldsymbol{A}$ has n distinct eigenvalues, $\boldsymbol{A}$ is diagonalizable and we can write

$$\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$$

since $\boldsymbol{V} = [\boldsymbol{v}_1 \quad \boldsymbol{v}_2 \quad \dots \quad \boldsymbol{v}_n]$ is invertible in this case.

In this case, computing $\exp\{t\boldsymbol{A}\}$ and $\boldsymbol{A}^k$ is "easy". The main difficulty is finding the eigenvalues (finding the roots of a degree n polynomial).

What if $\boldsymbol{A}$ does not have n distinct eigenvalues? Does this mean that $\boldsymbol{A}$ is not diagonalizable?

Example

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

To determine the eigenvalues, we need to compute the roots of the characteristic polynomial, i.e. solve $\det(\lambda I_3 - A) = 0$...

Algebraic Multiplicity of an Eigenvalue

We can always write the characteristic polynomial of $\boldsymbol{A}$ in terms of its roots, i.e.

$$\det(\lambda \boldsymbol{I}_n - \boldsymbol{A}) = (\lambda - \lambda_1)^{r_1}(\lambda - \lambda_2)^{r_2} \cdots (\lambda - \lambda_s)^{r_s}$$

where $\{\lambda_1, \dots, \lambda_s\}$ is the set of distinct eigenvalues of $\boldsymbol{A}$ with $1 \leq s \leq n$.

Definition

The algebraic multiplicity of the eigenvalue λ_j of the matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is the number of times the root λ_j appears in the characteristic polynomial of $\boldsymbol{A}$ and is denoted as r_j.

Eigenspace and Geometric Multiplicity of an Eigenvalue

Definition

Given $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, if $\lambda_0 \in \mathbb{C}$ is an eigenvalue of $\boldsymbol{A}$, then the eigenspace corresponding with λ_0, denoted as $\mathcal{E}(\lambda_0)$, is the subspace of $\mathbb{C}^n$ spanned by the eigenvectors corresponding to the eigenvalue λ_0, i.e.

$$\mathcal{E}(\lambda_0) = \mathrm{null}(\boldsymbol{A} - \lambda_0 \boldsymbol{I}_n)$$

Definition

The geometric multiplicity of the eigenvalue λ_j of the matrix $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ is the dimension of the eigenspace of λ_j and is denoted as m_j, i.e.

$$m_j = \dim(\mathrm{null}(\boldsymbol{A} - \lambda_j \boldsymbol{I}_n))$$

Fact: For each $j \in \{1, \dots, s\}$, $1 \le m_j \le r_j$.

When is $\boldsymbol{A}$ Diagonalizable?

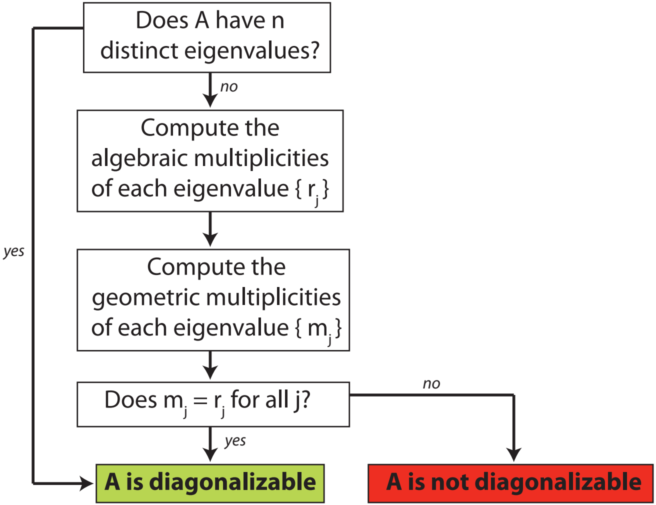
Theorem

If, for each $j \in \{1, \dots, s\}$, $m_j = r_j$, then $\boldsymbol{A}$ is diagonalizable.

This should be obvious when $\boldsymbol{A}$ has distinct eigenvalues since $m_j = r_j = 1$ for all j.

Proof sketch for the case when $\boldsymbol{A}$ does not have distinct eigenvalues:

A Procedure to Know When A is Diagonalizable



Summary of Diagonalization

1. Compute the eigenvalues of $\boldsymbol{A}$ and denote the distinct values as $\{\lambda_1, \dots, \lambda_s\}$.
2. If $\boldsymbol{A}$ is diagonalizable (see procedure on previous slide), then for each $j \in \{1, \dots, s\}$, find a basis for the eigenspace $\mathcal{E}(\lambda_j) = \mathrm{null}(\boldsymbol{A} - \lambda_j \boldsymbol{I}_n)$. You can do this with Gaussian elimination and echelon form. Let

$$B_j = \{\boldsymbol{v}_{j1}, \boldsymbol{v}_{j2}, \dots, \boldsymbol{v}_{jr_j}\}$$

   be a basis for $\mathcal{E}(\lambda_j)$.
3. Form $\boldsymbol{V}$ by stringing bases together. Note that $\boldsymbol{V}$ will be a square matrix since $\sum_{j=1}^{s} r_j = \sum_{j=1}^{s} m_j = n$.
4. Now $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{\Lambda}\boldsymbol{V}^{-1}$ (you should check it to be sure).

What Can We Do If $\boldsymbol{A}$ is Not Diagonalizable?

Some options:

1. It might be possible to just compute $\exp\{t\boldsymbol{A}\}$ by the definition, e.g.

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

2. You can use the fundamental matrix method to compute the CT-STM $\boldsymbol{\Phi}(t, s)$, and hence compute $\exp\{(t-s)\boldsymbol{A}\}$.
3. You can still use the eigenvalue/eigenvector method except you have to work with "generalized eigenvectors".

Generalized Eigenvectors

Definition

Given $\boldsymbol{A} \in \mathbb{R}^{n \times n}$ and $\lambda_0 \in \mathbb{C}$ an eigenvalue of $\boldsymbol{A}$, we say that $\boldsymbol{v}_0 \in \mathbb{C}^n$ is a generalized eigenvector corresponding with λ_0 if $\boldsymbol{v}_0 \neq \mathbf{0}$ and

$$(\boldsymbol{A} - \lambda_0 \boldsymbol{I}_n)^k \boldsymbol{v}_0 = \mathbf{0}$$

for some integer $k \geq 1$.

Question: Are all regular eigenvectors also generalized eigenvectors?

Question: Are all generalized eigenvectors also regular eigenvectors?

Generalized Eigenspace

Definition

The generalized eigenspace of the eigenvalue λ_0 is the subspace of $\boldsymbol{C}^n$ spanned by all of the generalized eigenvalues corresponding to λ_0.

We will use the notation $\mathcal{F}(\lambda_0)$ to denote the generalized eigenspace of the eigenvector λ_0.

Examples...

Some Basic Properties of Generalized Eigenspaces

1. $\mathcal{E}(\lambda_0) \subset \mathcal{F}(\lambda_0)$, i.e., the regular eigenspace of the eigenvalue λ_0 is a subset of the generalized eigenspace of the eigenvalue λ_0. Why?
2. If $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is a set of generalized eigenvectors corresponding to different eigenvalues, then $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_k\}$ is a linearly independent set.
3. If $\boldsymbol{v}_0 \in \mathcal{F}(\lambda_0)$, then $\boldsymbol{A}\boldsymbol{v}_0 \in \mathcal{F}(\lambda_0)$. In other words, the subspace $\mathcal{F}(\lambda_0)$ is invariant under $\boldsymbol{A}$.
4. Given the characteristic polynomial of $\boldsymbol{A}$

$$\det(\lambda \boldsymbol{I}_n - \boldsymbol{A}) = (\lambda - \lambda_1)^{r_1}(\lambda - \lambda_2)^{r_2} \cdots (\lambda - \lambda_s)^{r_s}$$

where $\lambda_1, \dots, \lambda_s$ are all distinct and $r_1, \dots, r_s$ are the respective algebraic multiplicities, it can be shown that

$$\dim(\mathcal{F}(\lambda_j)) = r_j.$$

When combined with property #1, this implies that

$$1 \le \dim(\mathcal{E}(\lambda_j)) \le \dim(\mathcal{F}(\lambda_j)) = r_j.$$

Some Basic Properties of Generalized Eigenspaces (cont.)

5. A consequence of properties #2 and #4. If

$$\begin{array}{ccc} \{\boldsymbol{v}_{11},\dots,\boldsymbol{v}_{1r_1}\} & \text{is a basis for} & \mathcal{F}(\lambda_1) \\ \vdots & \vdots & \vdots \\ \{\boldsymbol{v}_{s1},\dots,\boldsymbol{v}_{1r_s}\} & \text{is a basis for} & \mathcal{F}(\lambda_s) \end{array}$$

then we can string all of these sets of generalized eigenvectors together into a big set $\{\boldsymbol{v}_{11},\dots,\boldsymbol{v}_{1r_1},\dots,\boldsymbol{v}_{s1},\dots,\boldsymbol{v}_{1r_s}\}$.

How many vectors are in this set? ______

This set of generalized eigenvectors is a basis for $\boldsymbol{C}^n$. Why?

Some Basic Properties of Generalized Eigenspaces (cont.)

6. From property #3, if $\boldsymbol{v}_{jk} \in \mathcal{F}(\lambda_j)$, then $\boldsymbol{A}\boldsymbol{v}_{jk} \in \mathcal{F}(\lambda_j)$.

   $\Leftrightarrow$ $\boldsymbol{A}\boldsymbol{v}_{jk}$ can be expressed as a linear combination of the vectors comprising a basis for $\mathcal{F}(\lambda_j)$.

   $\Leftrightarrow$ If the basis for $\mathcal{F}(\lambda_j)$ is $\{\boldsymbol{v}_{j1}, \dots, \boldsymbol{v}_{jr_j}\}$ then

$$\begin{aligned} \boldsymbol{A}\boldsymbol{v}_{j1} &= \alpha_{11}\boldsymbol{v}_{j1} + \cdots + \alpha_{1r_1}\boldsymbol{v}_{jr_j} \\ &\vdots \\ \boldsymbol{A}\boldsymbol{v}_{jr_j} &= \alpha_{j1}\boldsymbol{v}_{j1} + \cdots + \alpha_{jr_1}\boldsymbol{v}_{jr_j} \end{aligned}$$

   $\Leftrightarrow$ We can rewrite these r_j equations as one big matrix equation:

$$\boldsymbol{A}\underbrace{\begin{bmatrix}\boldsymbol{v}_{j1} & \dots & \boldsymbol{v}_{jr_j}\end{bmatrix}}_{\boldsymbol{V}_j} = \underbrace{\begin{bmatrix}\boldsymbol{v}_{j1} & \dots & \boldsymbol{v}_{jr_j}\end{bmatrix}}_{\boldsymbol{V}_j}\underbrace{\begin{bmatrix}\alpha_{11} & \alpha_{21} & \dots & \alpha_{r_j1} \\ \alpha_{12} & \alpha_{22} & \dots & \alpha_{r_j2} \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_{1r_j} & \alpha_{2r_j} & \dots & \alpha_{r_jr_j}\end{bmatrix}}_{\boldsymbol{Q}_j}$$

Some Basic Properties of Generalized Eigenspaces (cont.)

Property #6 continued...

We now have $\boldsymbol{A}\boldsymbol{V}_j = \boldsymbol{V}_j\boldsymbol{Q}_j$. What are the dimensions of $\boldsymbol{A}$, $\boldsymbol{V}_j$, and $\boldsymbol{Q}_j$?

Let $\boldsymbol{V} = [\boldsymbol{V}_1 \quad \boldsymbol{V}_2 \quad \dots \quad \boldsymbol{V}_s]$ and

$$\boldsymbol{Q} = \begin{bmatrix} \boldsymbol{Q}_1 & & \\ & \ddots & \\ & & \boldsymbol{Q}_s \end{bmatrix} \text{ (block diagonal form)}$$

What are the dimensions of $\boldsymbol{V}$ and $\boldsymbol{Q}$?

We now have $\boldsymbol{A}\boldsymbol{V} = \boldsymbol{V}\boldsymbol{Q}$. From property #5, what can we say about the invertibility of $\boldsymbol{V}$?

Hence, we can write $\boldsymbol{A} = \boldsymbol{V}\boldsymbol{Q}\boldsymbol{V}^{-1}$. Note that $\boldsymbol{Q}$ is not diagonal, but block diagonal.

Some Basic Properties of Generalized Eigenspaces (cont.)

7. By the definition of generalized eigenvectors and generalized eigenspaces, the statement $\boldsymbol{v} \in \mathcal{F}(\lambda_j)$ is equivalent to

$$(\boldsymbol{A} - \lambda_j \boldsymbol{I}_n)^k \boldsymbol{v} = \boldsymbol{0} \quad (1)$$

for some integer $k \geq 1$. Note that, if (1) is true when $k = k_0$, then it is also true for all $k \geq k_0$.

This implies that

$$\mathcal{F}(\lambda_j) = \text{null}\left((\boldsymbol{A} - \lambda_j \boldsymbol{I}_n)^{r_j}\right).$$

In other words, to determine the generalized eigenspace for the eigenvalue λ_j, you don't need to compute the nullspace of $(\boldsymbol{A} - \lambda_j \boldsymbol{I}_n)^k$ for $k = 1, 2, \ldots, r_j$.

You can just compute the nullspace of the matrix $(\boldsymbol{A} - \lambda_j \boldsymbol{I}_n)^{r_j}$ by doing the standard Gaussian elimination and putting the result in echelon form.

Some Basic Properties of Generalized Eigenspaces (cont.)

8. From properties #6 and #7, we can say that

$$(\boldsymbol{Q}_j - \lambda_j \boldsymbol{I}_{r_j})^{r_j} = \boldsymbol{0}$$

Why?

Nilpotent Matrices

Definition

A nilpotent matrix is a square matrix $\hat{\boldsymbol{N}}$ with the property that $\hat{\boldsymbol{N}}^m = \mathbf{0}$ for some positive integer m.

An equivalent definition for a nilpotent matrix is a square matrix with eigenvalues all equal to zero.

Examples: Which of these matrices are

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \boldsymbol{A}_2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \qquad \boldsymbol{A}_3 = \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

nilpotent?

Some Basic Properties of Generalized Eigenspaces (cont.)

9. From property #8 we know that

$$\begin{bmatrix} \boldsymbol{Q}_1 - \lambda_1 \boldsymbol{I}_{r_1} & & \\ & \ddots & \\ & & \boldsymbol{Q}_s - \lambda_s \boldsymbol{I}_{r_s} \end{bmatrix} = \hat{\boldsymbol{N}}$$

is nilpotent. If we let

$$\boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 \boldsymbol{I}_{r_1} & & \\ & \ddots & \\ & & \lambda_s \boldsymbol{I}_{r_s} \end{bmatrix}$$

then we have $\boldsymbol{Q} = \boldsymbol{\Lambda} + \hat{\boldsymbol{N}}$ where Λ is diagonal and $\hat{\boldsymbol{N}}$ is nilpotent.

We can show that

$$\boldsymbol{\Lambda}\hat{\boldsymbol{N}} = \hat{\boldsymbol{N}}\boldsymbol{\Lambda}$$

In other words, $\hat{\boldsymbol{N}}$ and $\boldsymbol{\Lambda}$ commute.

Computing $\boldsymbol{A}^k$ When $\boldsymbol{A}$ is Not Diagonalizable

We now know everything we need to compute $\boldsymbol{A}^k$ when $\boldsymbol{A}$ is not diagonalizable. In general, we can always write

$$\boldsymbol{A} = \boldsymbol{V}(\boldsymbol{\Lambda} + \hat{\mathbf{N}})\boldsymbol{V}^{-1}$$

This implies that

$$\boldsymbol{A}^k = \boldsymbol{V}(\boldsymbol{\Lambda} + \hat{\mathbf{N}})^k\boldsymbol{V}^{-1}$$

By the binomial expansion theorem and property #9, we can write

$$\boldsymbol{A}^k = \boldsymbol{V}\left[\sum_{j=0}^{k}\binom{k}{j}\boldsymbol{\Lambda}^{k-j}\hat{\boldsymbol{N}}^j\right]\boldsymbol{V}^{-1}$$

But $\hat{\boldsymbol{N}}$ is nilpotent. Hence, for $j \geq \max\{r_1, \dots, r_s\}$, $\hat{\boldsymbol{N}}^j = 0$.

Computing $\exp\{t\boldsymbol{A}\}$ When $\boldsymbol{A}$ is Not Diagonalizable

By property #9, we can write

$$\begin{aligned}\exp\{t\boldsymbol{A}\} &= \boldsymbol{V}\exp\{t(\boldsymbol{\Lambda}+\hat{\boldsymbol{N}})\}\boldsymbol{V}^{-1}\\ &= \boldsymbol{V}\exp\{t\boldsymbol{\Lambda}\}\exp\{t\hat{\boldsymbol{N}}\}\boldsymbol{V}^{-1}\end{aligned}$$

The term $\exp\{t\boldsymbol{\Lambda}\}$ is easy to compute because $\boldsymbol{\Lambda}$ is diagonal.

What about the term $\exp\{t\hat{\boldsymbol{N}}\}$? Look at the definition:

$$\exp\{t\hat{\boldsymbol{N}}\} = \sum_{k=0}^{\infty}\frac{t^k}{k!}\hat{\boldsymbol{N}}^k.$$

But $\hat{\boldsymbol{N}}$ is nilpotent. So the sum will only have a finite number of terms:

$$\exp\{t\hat{\boldsymbol{N}}\} = \sum_{k=0}^{\max\{r_1,\dots,r_s\}-1}\frac{t^k}{k!}\hat{\boldsymbol{N}}^k.$$

In typical cases, there are only a few terms to compute.

Examples

Putting it All Together

A procedure for finding $\boldsymbol{A}^k$ and/or $\exp\{t\boldsymbol{A}\}$ for arbitrary $\boldsymbol{A} \in \mathbb{R}^{n\times n}$:

1. Find all of the eigenvalues of $\boldsymbol{A}$. Usually you do this by computing the roots of the characteristic polynomial, i.e.

$$\det(\lambda \boldsymbol{I}_n - \boldsymbol{A}) \;=\; (\lambda-\lambda_1)^{r_1}(\lambda-\lambda_2)^{r_2}\cdots(\lambda-\lambda_s)^{r_s} = 0$$

2. For each $j \in \{1,\dots,s\}$, find a basis for $\mathcal{E}(\lambda_j) = \mathrm{null}(\boldsymbol{A} - \lambda_j \boldsymbol{I}_n)$.
   - If $\dim(\mathcal{E}(\lambda_j)) = r_j$ then good! Move on to next eigenvalue.
   - If $\dim(\mathcal{E}(\lambda_j)) > r_j$ then you've done something wrong.
   - If $\dim(\mathcal{E}(\lambda_j)) < r_j$ then you need to find a basis for the generalized eigenspace $\mathcal{F}(\lambda_j) = \mathrm{null}(\boldsymbol{A} - \lambda_j \boldsymbol{I}_n)^{r_j}$. This basis must contain r_j linearly independent vectors.
3. Form $\boldsymbol{V} \in \mathbb{C}^{n\times n}$ by stringing together all of the bases.
4. Compute $\boldsymbol{V}^{-1}\boldsymbol{A}\boldsymbol{V} = \boldsymbol{Q} = \boldsymbol{\Lambda} + \hat{\boldsymbol{N}}$ where $\boldsymbol{\Lambda}$ is diagonal and $\hat{\boldsymbol{N}}$ is nilpotent. Note that $\hat{\boldsymbol{N}} = \boldsymbol{0}$ when $\boldsymbol{A}$ is diagonalizable.

Putting it All Together (cont.)

5. Compute $\boldsymbol{A}^k$ via

$$\boldsymbol{A}^k \;=\; \boldsymbol{V}\left[\sum_{j=0}^{k}\binom{k}{j}\boldsymbol{\Lambda}^{k-j}\hat{\boldsymbol{N}}^j\right]\boldsymbol{V}^{-1}$$

where the nilpotent property of $\hat{\boldsymbol{N}}$ implies that the sum will have at most $\max\{r_1,\dots,r_s\}-1$ terms for any k.

6. Compute $\exp\{t\boldsymbol{A}\}$ via

$$\exp\{t\boldsymbol{A}\}\boldsymbol{V}\exp\{t\boldsymbol{\Lambda}\}\exp\{t\hat{\boldsymbol{N}}\}\boldsymbol{V}^{-1}$$

where the term $\exp\{t\boldsymbol{\Lambda}\}$ is easy to compute because $\boldsymbol{\Lambda}$ is diagonal and the term

$$\exp\{t\hat{\boldsymbol{N}}\} \;=\; \sum_{k=0}^{\max\{r_1,\dots,r_s\}-1}\frac{t^k}{k!}\hat{\boldsymbol{N}}^k.$$

is also not too difficult since the sum is finite.

Remarks

Note that $\exp\{t\mathbf{\Lambda}\}$ will have elements that look like $e^{\lambda_1 t}, e^{\lambda_2 t}$,.... What will the elements of $\exp\{t\hat{\boldsymbol{N}}\}$ look like?

Hence, when $\boldsymbol{A}$ is diagonalizable, $\exp\{t\boldsymbol{A}\}$ will only have terms that look like $e^{\lambda t}$. When $\boldsymbol{A}$ is not diagonalizable, $\exp\{t\boldsymbol{A}\}$ will also have terms that look like $t^m e^{\lambda t}$.

Conclusions

This concludes our **quantitative** analysis of systems. We will be moving on to **qualitative** analysis (e.g. stability) after the midterm exam.

1. Solution to LTI or LTV discrete-time state-space difference equations (existence and uniqueness).
2. Solution to LTI or LTV continuous-time state-space differential equations (existence and uniqueness).
3. Important special case: DT LTI systems with $\boldsymbol{\Phi}[k,j] = \boldsymbol{A}^{k-j}$ and CT LTI systems with $\boldsymbol{\Phi}(t,s) = \exp\{(t-s)\boldsymbol{A}\}$.
4. Linear algebraic tools:
   - Subspaces
   - Nullspace, range, rank, nullity
   - Matrix invertibility
   - Eigenvalues, eigenvectors, eigenspaces, and nilpotent matrices
5. Properties of $\boldsymbol{A}^k$ and $\exp\{t\boldsymbol{A}\}$.
6. Computation of $\boldsymbol{A}^k$ and $\exp\{t\boldsymbol{A}\}$ when $\boldsymbol{A}$ is diagonalizable.
7. Computation of $\boldsymbol{A}^k$ and $\exp\{t\boldsymbol{A}\}$ when $\boldsymbol{A}$ is not diagonalizable.