ECE504: Lecture 8

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28-Oct-2008

Lecture 8 Major Topics

We are still in Part II of ECE504: **Quantitative and qualitative analysis of systems**

mathematical description $\rightarrow$ results about behavior of system

We will cover the following **qualitative** properties of systems:

- Stability
- Reachability and controllability
- Observability
- Minimality

Today, our focus is on analyzing the internal stability of LTI systems. We will also discuss techniques for analyzing the stability of nonlinear and time-varying systems.

You should be reading Chen Chapter 5 now. Sections 5.1 and 5.3-5.5 all discuss internal stability.

Stability: Some Intuition

Suppose you have a CT-LTI system described by

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \end{bmatrix} \boldsymbol{u}(t)\end{aligned}$$

Is this system "stable"?

Since this is an LTI system, we can compute the transfer function using our standard technique:

$$\hat{g}(s) = \boldsymbol{C}(s\boldsymbol{I}_3 - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D} = \frac{(s-1)^2}{(s+1)(s-1)^2} = \frac{1}{s+1}$$

Is this system stable?

Internal Versus External Stability

There are two types of stability that we can discuss when we use state-space descriptions of dynamic systems:

1. Internal stability (are the states blowing up?)
2. External stability (is the output blowing up?)

Transfer functions can only tell you about the external stability of systems. State-space descriptions can tell you about both internal and external stability. This is another big advantage of the state-space mathematical description with respect to transfer functions.

As we've seen, it is possible for systems to have external stability but not internal stability.

Bounded Vector-Valued Functions

Definition

The vector-valued function $\boldsymbol{z}(t) : \mathbb{R} \mapsto \mathbb{R}^n$ is **bounded** if there exists some finite $0 \leq M < \infty$ such that $\|\boldsymbol{z}(t)\| \leq M$ for all $t \in \mathbb{R}$.

Recall that

$$\begin{aligned} \|\boldsymbol{z}(t)\| &:= \sqrt{\boldsymbol{z}^\top(t)\boldsymbol{z}(t)} \\ &= \sqrt{z_1^2(t) + \cdots + z_n^2(t)} \end{aligned}$$

is the Euclidean norm.

Internal Stability of Continuous-Time Systems

Definition

A continuous-time system is **stable** if and only if, when the input $\boldsymbol{u}(t) \equiv 0$ for all $t \geq t_0$, the state $\boldsymbol{x}(t)$ is bounded for all $t \geq t_0$ for any initial state $\boldsymbol{x}(t_0) \in \mathbb{R}^n$.

Definition

A continuous-time system is **asymptotically stable** if and only if it is stable and

$$\lim_{t \to \infty} \|\boldsymbol{x}(t)\| = 0$$

for any initial state $\boldsymbol{x}(t_0) \in \mathbb{R}^n$.

Note: Our "stable" is Chen's "marginally stable".

Continuous-Time Nonlinear System Example

Suppose we had a system with scalar state dynamics

$$\dot{x}(t) \quad = \quad x(t)(1 - x^2(t))$$

1. What are the equilibria of this differential equation?
2. Is this system (internally) stable?
3. Is this system (internally) asymptotically stable?

Internal Stability of CT-LTI Systems

Continuous time LTI system:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)\end{aligned}$$

Note that internal stability has nothing to do with the output equation $\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$ or the matrix $\boldsymbol{B}$. Internal stability is all about the properties of the differential equation

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) \tag{1}$$

What are the equilibria of (1)?

Note that it is impossible to have multiple isolated equilibria when the system is LTI.

Remarks on Internal Stability

- If you are looking at a linear system, we can focus our attention on the equilibrium point $\boldsymbol{x}(t) = 0$. We don't need to consider other equilibria in the nullspace of $\boldsymbol{A}$ because their behavior will be the same as the equilibrium point $\boldsymbol{x}(t) = 0$.
- If you are looking at a nonlinear system, the behavior at each equilibrium point may be different. Each must be analyzed individually.
- Our definition of asymptotic (internal) stability is appropriate for linear systems because of the implicit equilibrium at the origin.
- Asymptotic (internal) stability may be defined differently for nonlinear systems.

Stability Criterion

Theorem

A continuous-time LTI system is (internally) stable if and only if both of the following conditions are true.

1. $\mathrm{Re}(\lambda_j) \le 0$ *for all* $j \in \{1, \dots, s\}$ *where* $\{\lambda_1, \dots, \lambda_s\}$ *is the set of distinct eigenvalues of* $\boldsymbol{A}$.
2. *For each* λ_j *that is an eigenvalue of* $\boldsymbol{A}$ *such that* $\mathrm{Re}(\lambda_j) = 0$, *the algebraic multiplicity of* λ_j *is equal to the geometric multiplicity of* λ_j, *i.e.* $r_j = m_j$.

Note that condition 2 is equivalent to $\boldsymbol{A}$ being "diagonalizable for all eigenvalues with zero real part". Chen Theorem 5.4 requires each eigenvalue with zero real part to have algebraic multiplicity one. It should be clear that Chen's condition is actually too strong.

Examples...

Stability Criterion Proof Sketch

Asymptotic Stability Criterion

Theorem

A continuous-time LTI system is asymptotically (internally) stable if and only if $\mathrm{Re}(\lambda_j) < 0$ *for all* $j \in \{1, \dots, s\}$ *where* $\{\lambda_1, \dots, \lambda_s\}$ *is the set of distinct eigenvalues of* $\boldsymbol{A}$.

Remarks:

1. It is easier to check asymptotic stability than stability.
2. Special name for matrices with all eigenvalues satisfying $\mathrm{Re}(\lambda_j) < 0$: "Hurwitz" (the same Hurwitz as the Routh-Hurwitz stability criterion you may have seen in an undergraduate systems/controls course).
3. The proof of this theorem follows directly from the fact that

$$\lim_{t \to 0} t^m e^{\alpha t} = 0$$

for any integer m if $\mathrm{Re}(\alpha) < 0$.

Internal Stability of Discrete-Time Systems

The definitions of internal stability and asymptotic internal stability for continuous-time systems can be easily extended to discrete-time systems:

Definition

A discrete-time system is **stable** if and only if, when the input $\boldsymbol{u}[k] \equiv 0$ for all $k \geq k_0$, the state $\boldsymbol{x}[k]$ is bounded for all $k \geq k_0$ for any initial state $\boldsymbol{x}[k_0] \in \mathbb{R}^n$.

Definition

A discrete-time system is **asymptotically stable** if and only if it is stable and

$$\lim_{k \to \infty} \|\boldsymbol{x}[k]\| = 0$$

for any initial state $\boldsymbol{x}[k_0] \in \mathbb{R}^n$.

Stability Criterion

Theorem

A discrete-time LTI system is (internally) stable if and only if both of the following conditions are true.

1. $|\lambda_j| \le 1$ *for all* $j \in \{1, \dots, s\}$ *where* $\{\lambda_1, \dots, \lambda_s\}$ *is the set of distinct eigenvalues of* $\boldsymbol{A}$.
2. *For each* λ_j *that is an eigenvalue of* $\boldsymbol{A}$ *such that* $|\lambda_j| = 1$, *the algebraic multiplicity of* λ_j *is equal to the geometric multiplicity of* λ_j, *i.e.* $r_j = m_j$.

The proof of this theorem is similar to the proof for the continuous-time case except that the terms of $\boldsymbol{A}^k$ look like λ_j^k, $k\lambda_j^k$, $k^2\lambda_j^k$, etc.

- If $|\lambda_j| < 1$, then $k^m\lambda_j^k$ is always bounded.
- If $|\lambda_j| = 1$, then condition 2 implies that there are only λ_j^k terms, no $k^m\lambda_j^k$ terms.

Discrete-Time Asymptotic Stability Criterion

Theorem

A discrete-time LTI system is asymptotically (internally) stable if and only if $|\lambda_j| < 1$ for all $j \in \{1, \dots, s\}$ where $\{\lambda_1, \dots, \lambda_s\}$ is the set of distinct eigenvalues of $\boldsymbol{A}$.

Remarks:

1. Special name for matrices with all eigenvalues satisfying $|\lambda_j| < 1$: "Schur" (the same Schur as the Schur-Cohn stability criterion).
2. The proof of this theorem follows directly from the fact that

$$\lim_{k \to 0} k^m \lambda_j^k = 0$$

   for any integer m if $|\lambda_j| < 1$.

Discrete-Time Internal Stability Examples

Lyapunov Stability

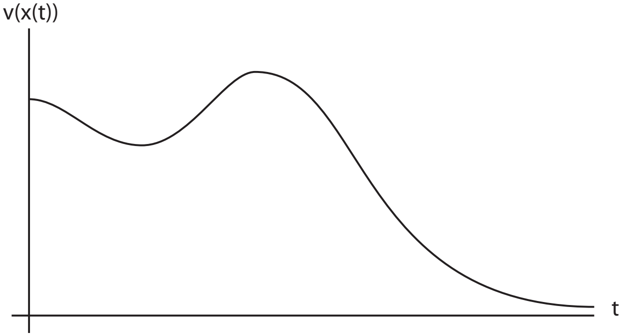
- Lyapunov stability is a general internal stability concept that is particularly useful for nonlinear differential/difference equations.
- The ideas of Lyapunov stability can also apply to LTI systems.

The main idea:

1. Assume the input $\boldsymbol{u}(t) \equiv 0$ (we are looking at internal stability).
2. Let $v(\boldsymbol{x}(t))$ be the squared "distance" of the state $\boldsymbol{x}(t)$ from the equilibrium of interest.
3. If $\dot{v}(\boldsymbol{x}(t)) < 0$ for all $\boldsymbol{x}(t)$ not at the equilibrium of interest, then the state $\boldsymbol{x}(t)$ must be getting closer to the equilibrium. Why?
4. If the "distance" to the equilibrium is strictly decreasing for all $\boldsymbol{x}(t)$, then the system is asymptotically stable.

Recall that nonlinear systems may have many different equilibria but, in LTI systems, we only need to consider the equilibrium $\boldsymbol{x}(t) = 0$.

Lyapunov Stability



Note that $v(\boldsymbol{x}(t)) \to 0$ (the system is asymptotically stable) even though this example shows a case when the "distance" to the equilibrium is not a strictly decreasing function of time. Hence, $\dot{v}(\boldsymbol{x}(t)) < 0$ for all $\boldsymbol{x}(t)$ implies asymptotic stability, but the converse is not always true.

An Example of Lyapunov Stability for Nonlinear Systems

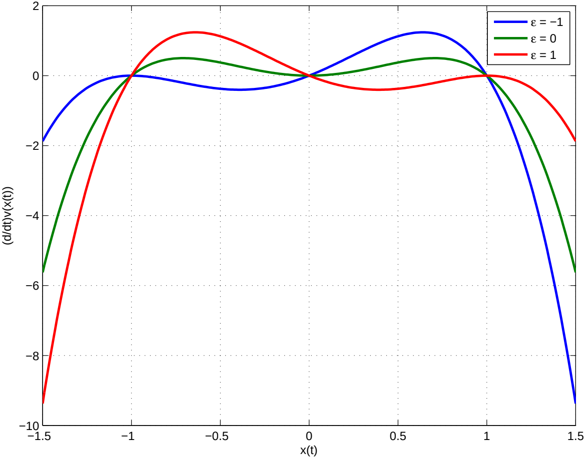
Let's look again at the continuous-time nonlinear scalar differential equation

$$\dot{x}(t) = x(t)(1 - x^2(t))$$

We know of three equilibria: $x(t) \in \{-1, 0, 1\}$. Let's look at the Lyapunov stability of each of these equilibria...

An Example of Lyapunov Stability for Nonlinear Systems

$\dot{x}(t) = x(t)(1 - x^2(t))$ and $\dot{v}(x(t)) = (x(t) - \epsilon)^2$ for $\epsilon \in \{-1, 0, 1\}$



None of the equilibria meet the requirements for Lyapunov stability. Why?

Positive Definite and Positive Semi-Definite Matrices

Definition

A matrix $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ is **positive definite** if $\boldsymbol{P} = \boldsymbol{P}^\top$ and $\boldsymbol{P}$ has all positive eigenvalues.

Remarks:

1. When $\boldsymbol{P} = \boldsymbol{P}^\top$, the matrix $\boldsymbol{P}$ is said to be symmetric.
2. Fact: All of the eigenvalues of real-valued symmetric matrices are real.
3. Note: Chen's definition of positive definiteness does not require symmetry (his theorems say "symmetric and positive definite").

Definition

A matrix $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ is **positive semi-definite** if $\boldsymbol{P} = \boldsymbol{P}^\top$ and $\boldsymbol{P}$ has all non-negative eigenvalues.

Obviously, any positive definite $\boldsymbol{P}$ is also positive semi-definite, but the converse is not always true.

Quadratic Form

In linear algebra, we call expressions like $\boldsymbol{z}^\top \boldsymbol{P} \boldsymbol{z}$ the "quadratic form".

- For any $\boldsymbol{P} \in \mathbb{R}^{n \times n}$, $\boldsymbol{z} \in \mathbb{R}^n$, and any integer $n \geq 1$, the result of the quadratic form is just a scalar.
- If $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ is positive definite, then it is not too difficult to show that $\boldsymbol{z}^\top \boldsymbol{P} \boldsymbol{z} > 0$ for all $\boldsymbol{z} \in \mathbb{R}^n$ and $\boldsymbol{z} \neq 0$.
- Similarly, if $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ is positive semi-definite, then we can show that $\boldsymbol{z}^\top \boldsymbol{P} \boldsymbol{z} \geq 0$ for all $\boldsymbol{z} \in \mathbb{R}^n$.

Relationship to Euclidean norm:

- Recall that $\|\boldsymbol{z}\|^2 := z_1^2 + \cdots + z_n^2 = \boldsymbol{z}^\top \boldsymbol{z} = \boldsymbol{z}^\top \boldsymbol{I}_n \boldsymbol{z}$.
- Is $\boldsymbol{I}_n$ positive definite?
- For positive definite $\boldsymbol{P}$, we can define a generalized norm as $\|\boldsymbol{z}\|_{\boldsymbol{P}}^2 := \boldsymbol{z}^\top \boldsymbol{P} \boldsymbol{z}$. Note that, like the Euclidean norm, this generalized norm satisfies all of the intuitive properties of a distance measure.

Lyapunov Matrix Equation: CT-LTI Systems

Let $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ be a constant positive definite matrix and define

$$v(\boldsymbol{x}(t)) \quad := \quad \boldsymbol{x}^\top(t)\boldsymbol{P}\boldsymbol{x}(t)$$

Assuming a continuous-time LTI system, we can write

$$\begin{aligned}
\dot{v}(\boldsymbol{x}(t)) &= \left(\frac{d}{dt}\boldsymbol{x}^\top(t)\right)\boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{x}^\top(t)\boldsymbol{P}\left(\frac{d}{dt}\boldsymbol{x}(t)\right) \\
&= \left(\frac{d}{dt}\boldsymbol{x}(t)\right)^\top \boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{x}^\top(t)\boldsymbol{P}\left(\frac{d}{dt}\boldsymbol{x}(t)\right) \\
&= (\boldsymbol{A}\boldsymbol{x}(t))^\top \boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{x}^\top(t)\boldsymbol{P}(\boldsymbol{A}\boldsymbol{x}(t)) \\
&= \boldsymbol{x}^\top(t)\boldsymbol{A}^\top\boldsymbol{P}\boldsymbol{x}(t) + \boldsymbol{x}^\top(t)\boldsymbol{P}\boldsymbol{A}\boldsymbol{x}(t) \\
&= \boldsymbol{x}^\top(t)\left(\boldsymbol{A}^\top\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}\right)\boldsymbol{x}(t) \\
&= -\boldsymbol{x}^\top(t)\boldsymbol{Q}\boldsymbol{x}(t)
\end{aligned}$$

Is $\boldsymbol{Q} := -\left(\boldsymbol{A}^\top\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A}\right)$ is symmetric? When must $\dot{v}(\boldsymbol{x}(t)) < 0$?

Lyapunov Theorem: CT-LTI Systems

Recall that, for CT-LTI systems:

$\boldsymbol{A}$ is Hurwitz $\Leftrightarrow$ the CT-LTI system is asymptotically stable

Theorem

$\boldsymbol{A}$ is Hurwitz if and only if, for any positive definite $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$, there exists a unique positive definite $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ satisfying $\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{Q}$.

The proof of this theorem can be found in the Chen textbook Section 5.4.

The implied procedure:

1. You are given $\boldsymbol{A} \in \mathbb{R}^{n \times n}$.
2. You pick a particular $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$ that must be positive definite. A good choice is $\boldsymbol{I}_n$.
3. You then solve for $\boldsymbol{P} \in \mathbb{R}^{n \times n}$. This must be done element by element for each term in $\boldsymbol{P}$.

Lyapunov Theorem: CT-LTI Systems

Example:

$$\boldsymbol{A} = \begin{bmatrix} -1 & -1 \\ 0 & -1 \end{bmatrix}$$

Clearly, this $\boldsymbol{A}$ is Hurwitz. Let's try out the Lyapunov theorem...

Comments on the Lyapunov Theorem for CT-LTI Systems

- In general, solving for $\boldsymbol{P}$ in the equation $\boldsymbol{A}^\top \boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} = -\boldsymbol{Q}$ means that there are n^2 simultaneous linear equations to solve.
- The Lyapunov theorem has limited usefulness for low-dimension, e.g. $n = 2$ or $n = 3$, CT-LTI systems because it is usually easier to just compute the eigenvalues from the characteristic polynomial.
- For larger dimensional problems, e.g. $n \geq 5$, it might be easier to work with the Lyapunov theorem and the n^2 linear equations.
- The main ideas of Lyapunov stability are particularly useful in nonlinear systems, where the goal is to show that

$$\frac{d}{dt}\|\boldsymbol{x}(t) - \boldsymbol{\epsilon}\|^2 < 0$$

  for all $\boldsymbol{x}(t) \in \mathbb{R}^n$ where $\boldsymbol{\epsilon}$ is an equilibrium of the system.
- In nonlinear systems, **local** Lyapunov stability is also useful, i.e.

$$\frac{d}{dt}\|\boldsymbol{x}(t) - \boldsymbol{\epsilon}\|^2 < 0 \text{ for all } \boldsymbol{x}(t) \in \mathcal{X}_\epsilon \subset \mathbb{R}^n.$$

Lyapunov Stability: DT-LTI Systems

In DT-LTI systems, we only need to analyze the equilibrium at the origin, so the Lyapunov idea says that we want

$$v(\boldsymbol{x}[k+1]) - v(\boldsymbol{x}[k]) < 0$$

where

$$v(\boldsymbol{x}[k]) = \boldsymbol{x}^\top[k]\boldsymbol{P}\boldsymbol{x}[k] \text{ with positive definite } \boldsymbol{P}$$

is a generalized squared distance as in the CT-LTI case. We can write

$$\begin{aligned}
& v(\boldsymbol{x}[k+1]) - v(\boldsymbol{x}[k]) < 0 \\
\Leftrightarrow \quad & v(\boldsymbol{A}\boldsymbol{x}[k]) - v(\boldsymbol{x}[k]) < 0 \\
\Leftrightarrow \quad & \boldsymbol{x}^\top[k]\boldsymbol{A}^\top\boldsymbol{P}\boldsymbol{A}\boldsymbol{x}[k] - \boldsymbol{x}^\top[k]\boldsymbol{P}\boldsymbol{x}[k] < 0 \\
\Leftrightarrow \quad & \boldsymbol{x}^\top[k]\left(\boldsymbol{A}^\top\boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\right)\boldsymbol{x}[k] < 0 \\
\Leftrightarrow \quad & -\boldsymbol{x}^\top[k]\boldsymbol{Q}\boldsymbol{x}[k] < 0
\end{aligned}$$

Is $\boldsymbol{Q} := \boldsymbol{P} - \boldsymbol{A}^\top\boldsymbol{P}\boldsymbol{A}$ is symmetric? When must $v(\boldsymbol{x}[k+1]) - v(\boldsymbol{x}[k]) < 0$?

Lyapunov Theorem: DT-LTI Systems

Recall that, for DT-LTI systems:

$\boldsymbol{A}$ is Schur $\Leftrightarrow$ the DT-LTI system is asymptotically stable

Theorem

$\boldsymbol{A}$ is Schur if and only if, for any positive definite $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$, there exists a unique positive definite $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ satisfying $\boldsymbol{P} - \boldsymbol{A}^\top \boldsymbol{P} \boldsymbol{A} = \boldsymbol{Q}$.

The proof of this theorem can also be found in the Chen textbook Section 5.4.

The same basic procedure as the CT-LTI case applies here (in general, you have to solve n^2 simultaneous linear equations).

Stability of Time-Varying Systems

- Example

$$\boldsymbol{A}(t) = \begin{bmatrix} -1 & e^{2t} \\ 0 & -1 \end{bmatrix}$$

  What are the eigenvalues of $\boldsymbol{A}(t)$? Is this system stable or asymptotically stable?
- We can use the fundamental matrix method to compute the CT-STM:

$$\boldsymbol{\Phi}(t,0) = \frac{1}{2}\begin{bmatrix} 2e^{-t} & e^{t} - e^{-t} \\ 0 & 2e^{-t} \end{bmatrix}$$

  Now what do you think about the internal stability of this system?
- In general, eigenvalue analysis is not useful for time-varying systems. The concept of Lyapunov stability, however, can still be applied (but the theorems for LTI systems do not directly apply).
- Please read Chen pp. 138-140 for a discussion of the internal stability of time-varying systems.

Final Remarks

1. State-space description allows us to analyze the internal stability of dynamic systems.
2. General concepts of **stability** and **asymptotic stability** apply to linear, non-linear, time-invariant, and time-varying systems.
3. The concept of **local** stability applies only to nonlinear systems with distinct equilibria.
4. Several stability theorems for LTI systems.
5. Lyapunov stability analysis is a general tool that can be applied to all types of systems.