ECE504 Midterm Exam

21-Oct-2008

Notes:

- This exam is worth 350 points and is to be completed in 90 minutes.
- Look over all the questions before starting.
- Budget your time to allow enough time to work on each question.
- To receive maximum credit, you must show your reasoning and/or work.

1. 80 points total. Given the continuous time system shown in Figure 1, answer the following questions.

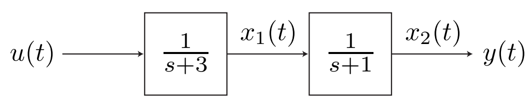

Figure 1: A continuous time system.

   (a) 10 pts. Classify this system as
      i. memoryless, lumped, or distributed
      ii. causal or noncausal
      iii. linear or nonlinear
      iv. time varying or time invariant

   (b) 30 pts. Defining the state as $\boldsymbol{x}(t) = [x_1(t), x_2(t)]^\top$ with $x_1(t)$ and $x_2(t)$ as shown in Figure 1, explicitly write a state-space realization of this system such that

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)u(t) \\ y(t) &= \boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{D}(t)u(t) \end{aligned}$$

   (c) 20 pts. Find the transfer function of this system. $\hat{g}(s) = \frac{\hat{y}(s)}{\hat{u}(s)}$.

   (d) 20 pts. Find a different state-space realization for this system that has the same transfer function.

2. 80 points total. Given a continuous-time state-space description

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \begin{bmatrix} -3 & 0 \\ 1 & -1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t) \\ y(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \boldsymbol{x}(t) \end{aligned}$$

   with initial state $\boldsymbol{x}(0) = [1, 1]^\top$ and input $u(t) = 0$ for all $t \in \mathbb{R}$, write a general expression for the output $y(t)$.

3. 90 points total. You are given the following input-output description of a discrete time system:

$$y[k] = ky[k-1] + u[k-1].$$

(a) 10 pts. Classify this system as

i. memoryless, lumped, or distributed

ii. causal or noncausal

iii. linear or nonlinear

iv. time varying or time invariant

(b) 30 pts. Using any reasonable choice for the state $\boldsymbol{x}(k)$, explicitly write a state-space realization of this system such that

$$\begin{aligned} \boldsymbol{x}[k+1] &= \boldsymbol{A}[k]\boldsymbol{x}[k] + \boldsymbol{B}[k]u[k] \\ y[k] &= \boldsymbol{C}[k]\boldsymbol{x}[k] + \boldsymbol{D}[k]u[k] \end{aligned}$$

(c) 50 pts. Find an explicit solution to this system that expresses $y[k]$ for all $k \geq k_0$ in terms of the given initial state $\boldsymbol{x}[k_0]$ and the input $u[k]$ for $k \geq k_0$.

4. 100 points total. Given

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix}$$

where $a \in \mathbb{R}$ and $b \in \mathbb{R}$.

(a) 20 pts. For the special case $a = 0$ and $b = 0$, compute expressions for $e^{t\boldsymbol{A}}$ and $\boldsymbol{A}^k$.

(b) 40 pts. For $a \in \mathbb{R}$ and $b \in \mathbb{R}$, compute a general expression for $e^{t\boldsymbol{A}}$.

(c) 40 pts. For $a \in \mathbb{R}$ and $b \in \mathbb{R}$, compute a general expression for $\boldsymbol{A}^{100}$.
Hint: The binomial expansion might be useful here. Recall that, given $\boldsymbol{P} \in \mathbb{R}^{n \times n}$ and $\boldsymbol{Q} \in \mathbb{R}^{n \times n}$ such that $\boldsymbol{P}$ and $\boldsymbol{Q}$ commute,

$$(\boldsymbol{P} + \boldsymbol{Q})^k = \sum_{m=0}^{k} \begin{pmatrix} k \\ m \end{pmatrix} \boldsymbol{P}^m \boldsymbol{Q}^{k-m} \tag{1}$$

where

$$\begin{pmatrix} k \\ m \end{pmatrix} = \frac{k!}{m!(k-m)!}. \tag{2}$$

Also recall that $0! = 1$.