1. Here is an invertible matrix that is not diagonalizable.

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad A^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$$

This matrix has only one distinct eigenvalue $\lambda_1 = 1$
- algebraic multiplicity $= 2$
- geometric multiplicity $= 1$ since

$$\text{null}(\lambda_1 I_2 - A) = \text{null}\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \qquad \text{basis} = \left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}$$

2. Here is an example of a diagonalizable matrix that is not invertible

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \leftarrow \text{already diagonal}$$

clearly not invertible since $\det(A) = 0$.

3.

$$A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}, \text{ characteristic polynomial } = (\lambda - a)^2 + b^2$$
$$= \lambda^2 - 2a\lambda + (a^2 + b^2)$$

roots from quadratic equation: $\dfrac{2a \pm \sqrt{4a^2 - 4(a^2+b^2)}}{2}$

hence $\lambda_1 = a + jb$, $\lambda_2 = a - jb$ (complex conjugate pair)

find e-vectors...

$$A - \lambda_1 I_2 = \begin{bmatrix} -jb & b \\ -b & -jb \end{bmatrix}, \quad v_1 = \begin{bmatrix} 1 \\ j \end{bmatrix}$$

$$A - \lambda_2 I_2 = \begin{bmatrix} jb & b \\ -b & jb \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ -j \end{bmatrix}$$

hence
$$e^{At} = \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} e^{(a+jb)t} & 0 \\ 0 & e^{(a-jb)t} \end{bmatrix} \begin{bmatrix} 1/2 & -j/2 \\ 1/2 & j/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ j & -j \end{bmatrix} \begin{bmatrix} \frac{1}{2} e^{(a+jb)t} & -\frac{j}{2} e^{(a+jb)t} \\ \frac{1}{2} e^{(a-jb)t} & \frac{j}{2} e^{(a-jb)t} \end{bmatrix}$$

continued...

$$= \begin{bmatrix} \frac{1}{2}e^{(a+jb)t} + \frac{1}{2}e^{(a-jb)t} & -\frac{j}{2}e^{(a+jb)t} + \frac{j}{2}e^{(a-jb)t} \\ \frac{j}{2}e^{(a+jb)t} - \frac{j}{2}e^{(a-jb)t} & \frac{1}{2}e^{(a+jb)t} + \frac{1}{2}e^{(a-jb)t} \end{bmatrix}$$

factor out e^{at}, use Euler's identities...

$$= e^{at}\begin{bmatrix} \cos bt & \sin at \\ -\sin at & \cos bt \end{bmatrix} = e^{At}.$$

4. a) Given $A = \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 1 \\ 0 & 0 & 1 \end{bmatrix}$, compute $\det(\lambda I_3 - A)$

$$= \lambda(\lambda+1)(\lambda-1) \qquad \lambda_1 = 0,\ \lambda_2 = -1,\ \lambda_3 = 1$$

This system can't be asymptotically stable because $\lambda_3 = 1$. It also can't be internally stable.

To look at BIBO stability, let's compute the transfer fn..

$$\hat{g}(s) = C(sI_3 - A)^{-1}B$$

$$(sI_3 - A)^{-1} = \begin{bmatrix} \frac{1}{s+1} & 0 & \frac{2}{(s+1)(s-1)} \\ 0 & 1/s & \frac{1}{s(s-1)} \\ 0 & 0 & \frac{1}{s-1} \end{bmatrix}$$

(you can get this from the adjoint)

Hence $\hat{g}(s) = [1\ 0\ 0]\begin{bmatrix} & & \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{s+1}$

The pole is at $s = -1$, so the system is BIBO stable.

b) We can make a minimal realization directly from the transfer function using our controllable canonical form

$$\dot{x}(t) = -x(t) + u(t)$$

$$y(t) = x(t) + 0$$

check: $C(sI_1 - A)^{-1}B + D = 1 \cdot (s+1)^{-1} \cdot 1 + 0 = \frac{1}{s+1}$ ✓

This is a minimal system (easy to check reachabily/observability).

This system is asymptotically stable (internally stable) and BIBO stable (as a consequence of asymtotic stability).

5. Given $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 1 \end{bmatrix}$ and $D = 1$

let's compute the transfer function...

$$\hat{g}(s) = C\,(sI_2 - A)^{-1}B + D = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s-1 & -1 \\ -1 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + 1$$

$$= \frac{1}{(s-1)^2 - 1} \left\{ \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s-1 & 1 \\ 1 & s-1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} + 1$$

$$= \frac{1}{s^2 - 2s} \left\{ \begin{bmatrix} s & s \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\} + 1 \quad = 1$$

so $\hat{g}(s) = 1 \Rightarrow y(t) = u(t)$ no dynamics!

The McMillan degree of this system is $\underline{n = 0}$.

6. Discrete-time system

$$x[k+1] = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} x[k] + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u[k]$$

a) To find the set of reachable states, we can look at the range of the reachability matrix

$$Q_r = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} b_1 & b_1 - b_2 \\ b_2 & b_2 - b_1 \end{bmatrix}$$

The set of reachable states will be all of $\mathbb{R}^2$ if $\text{range}(Q_r) = \mathbb{R}^2$, which is equivalent to $\det(Q_r) \neq 0$.

$$\det(Q_r) = b_1(b_2 - b_1) - b_2(b_1 - b_2) = b_2^2 - b_1^2$$

So this is a reachable system unless $b_2 = \pm b_1$

When $b_2 = b_1$ and $b_1 \neq 0$, $Q_r = \begin{bmatrix} b_1 & 0 \\ b_1 & 0 \end{bmatrix}$ and a basis for the set of reachable states is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

when $b_2 = -b_1$ and $b_1 \neq 0$, $Q_r = \begin{bmatrix} b_1 & 2b_1 \\ -b_1 & -2b_1 \end{bmatrix}$ and a basis for the set of reachable states is $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

Otherwise a basis for the set of reachable states is $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$

[note that if $b_1 = b_2 = 0$ then there are no reachable states]

b) The set of controllable states can only differ from the set of reachable states when the set of reachable states is not all of $\mathbb{R}^2$.

$$x[1] = Ax[0] + Bu[0]$$

$$x[2] = Ax[1] + Bu[1] = A^2 x[0] + ABu[0] + Bu[1] = 0$$

$$A^2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 2A$$

So we have $2Ax[0] + ABu[0] + Bu[1] = 0$ ← if we can find $u[0]$ and $u[1]$ so that this is true for an $x[0]$, then $x[0]$ is controllable.

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix} = \begin{bmatrix} b_1 - b_2 & b_1 \\ b_2 - b_1 & b_2 \end{bmatrix}\begin{bmatrix} u[0] \\ u[1] \end{bmatrix}$$

<u>when $b_2 = b_1$ and $b_1 \neq 0$</u> then
$$b_1 u[1] = 2x_1[0] - 2x_2[0]$$
$$b_1 u[1] = -2x_1[0] + 2x_2[0]$$

hence the state $x[0]$ is controllable only if $2x_1[0] - 2x_2[0] = -2x_1[0] + 2x_2[0]$
$\Longleftrightarrow x_1[0] = x_2[0]$ (same as reachable states)

when $b_2 = -b_1$ and $b_1 \neq 0$ then

$$2b_1 u[0] + b_1 u[1] = 2x_1[0] - 2x_2[0]$$

$$-2b_1 u[0] - b_1 u[1] = -2x_1[0] + 2x_2[0]$$

hence the state $x[0]$ is controllable only if

$$2x_1[0] - 2x_2[0] = -(-2x_1[0] + 2x_2[0])$$

which is true for any $x_1[0]$ and $x_2[0]$.

Hence, when $b_1 = -b_2$, the set of controllable states $= \mathbb{R}^2$ (this is different than the set of reachable states).

when $b_2 = b_1 = 0$ then

$$0 = 2x_1[0] - 2x_2[0]$$

$$0 = -2x_1[0] + 2x_2[0]$$

This is satisfied if $x_1[0] = x_2[0]$. Hence, a basis for the set of controllable states is $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ when $b_1 = b_2 = 0$.

7. a)

Let $u = r - [k_1 \; k_2]\, x$

Then $\dot{x} = \left(\begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ k_1 & k_2 \end{bmatrix} \right) x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} r$

$$= \underbrace{\begin{bmatrix} 2-k_1 & 3-k_2 \\ -k_1 & -1-k_2 \end{bmatrix}}_{A_f} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} r$$

$$|sI - A_f| = \begin{vmatrix} s-2+k_1 & k_2-3 \\ k_1 & s+k_2+1 \end{vmatrix}$$

$$= (s-2+k_1)(s+k_2+1) - k_1(k_2-3)$$

$$= s^2 + k_2 s + s - 2s - 2k_2 - 2 + k_1 s + \cancel{k_1 k_2} + k_1 - \cancel{k_1 k_2} + 3k_1$$

$$= s^2 + (k_2 - 1 + k_1)s + (4k_1 - 2k_2 - 2)$$

We want char. polynomial to be $(s+1)(s+2) = s^2 + 3s + 2$

Equating, $\quad k_2 - 1 + k_1 = 3 \quad (1)$

$\quad\quad 4k_1 - 2k_2 - 2 = 2 \quad (2)$

From (1), $\quad k_1 = 4 - k_2$

Sub k_1 into (2), $\quad 4(4-k_2) - 2k_2 - 2 = 2.$

$$16 - 4k_2 - 2k_2 = 4$$

$$6k_2 = 12 \Rightarrow k_2 = 2$$

$$\therefore k_1 = 4 - 2 = 2$$

$$\therefore \boxed{F = [2,\ 2]}$$

b) F is unique.

continued...

c)

$$A - BF = \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \end{bmatrix} \begin{bmatrix} 2 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}.$$

$$|\lambda I - (A-BF)| = \begin{vmatrix} \lambda & -1 \\ 2 & \lambda+3 \end{vmatrix} = \lambda(\lambda+3) + 2 = \lambda^2 + 3\lambda + 2$$
$$= (\lambda+2)(\lambda+1)$$

Since the e-values $\lambda_1 = -2$ and $\lambda_2 = -1$ all have negative real parts, the feedback system is asymptotically stable. ✓

Hence it is also BIBO stable. ✓

d)

$$Q_r = \begin{bmatrix} 1 & 1 \\ 1 & -5 \end{bmatrix} \rightarrow \text{rank } Q_r = 2 \Rightarrow$$ system with feedback is reachable ✓

e)

$$Q_o = \begin{bmatrix} C \\ C(A-BF) \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -2 & -2 \end{bmatrix} \rightarrow \text{rank } Q_o = 1 \Rightarrow$$ system with feedback not observable ✓

f)

Since the feedback system is not observable, it is not minimal.

(min. system must be both observable and reachable). ✓