Tips: Make sure your reasoning and work are clear to receive full credit for each problem.

1. 8 pts. Given the circuit in Figure 1,

\[ R_1 \quad R_2 \quad C \quad L \]
\[ v_1(t) \quad v_2(t) \]
\[ u(t) \quad + \quad - \]

Figure 1: A circuit.

(a) Let the state \( x(t) = [v_1(t) \ v_2(t)]^T \) and let the output \( y(t) \) be equal to the current through \( R_2 \). Find the matrices \( A(t), B(t), C(t), \) and \( D(t) \) such that

\[
\dot{x}(t) = A(t)x(t) + B(t)u(t) \\
y(t) = C(t)x(t) + D(t)u(t).
\]

(b) Write the input-output differential equation for this system, e.g. \( y(t) = \text{f}(\dot{y}(t), \ddot{y}(t), \ldots, u(t), \dot{u}(t), \ddot{u}(t), \ldots) \).

(c) Write the transfer function for this system.

(d) Classify this system:
   i. Memoryless, lumped, or distributed
   ii. Causal or noncausal
   iii. Linear or nonlinear
   iv. Time varying or time invariant

2. 4 pts. Chen Problem 2.6.

3. 4 pts. Qualitative description of systems:

   (a) Suppose we are given a system with two inputs and one output with an input-output relationship

\[
y(t) = \min(u_1(t), u_2(t)).
\]

Classify this system as in 1.(d).

(b) Suppose we have a discrete-time system with an input-output relationship

\[
y(k) = \frac{1}{1-\lambda} \sum_{m=0}^{\infty} \lambda^m u(k-m+1)
\]

where \( 0 < \lambda < 1 \) is known as the “forgetting factor”. Classify this system as in 1.(d). If \( u(k) = 1 \ \forall k \), what is \( y(k) \)?
4. 5 pts. Suppose we have a discrete time system that computes the moving average of a finite number of past inputs such that

\[ y(k) = \frac{1}{N} \sum_{n=0}^{N-1} u(k-n). \]

(a) Let \( N = 4 \) and let the state \( x(k) = [u(k-1) \ u(k-2) \ u(k-3)]^\top \). Find the matrices \( A(k) \), \( B(k) \), \( C(k) \), and \( D(k) \) such that

\[
x(k + 1) = A(k)x(k) + B(k)u(k) \\
y(k) = C(k)x(k) + D(k)u(k).
\]

(b) Let \( N = 4 \) and let the state

\[
x(k) = \begin{bmatrix} u(k-1) + u(k-2) + u(k-3) \\ u(k-1) + u(k-2) \\ u(k-1) \end{bmatrix}
\]

Find the matrices \( A(k) \), \( B(k) \), \( C(k) \), and \( D(k) \) such that

\[
x(k + 1) = A(k)x(k) + B(k)u(k) \\
y(k) = C(k)x(k) + D(k)u(k).
\]

(c) Comment on the uniqueness of the state.

5. 4 pts. Find the transfer function \( \hat{g}(s) = \frac{\hat{y}(s)}{\hat{u}(s)} \) of the system

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} x(t) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} c_1 & c_2 \end{bmatrix} x(t) + d u(t).
\end{align*}
\]

Simplify your answer to the form

\[
\hat{g}(s) = \frac{\lambda_2 s^2 + \lambda_1 s + \lambda_0}{\gamma_2 s^2 + \gamma_1 s + \gamma_0}.
\]

Hint: Recall that

\[
\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.
\]

6. 5 pts. Suppose you are given the 2-dimensional system described by the state space equations

\[
\begin{align*}
\dot{x}(t) &= \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t) \\
y(t) &= \begin{bmatrix} 1 & 1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \end{bmatrix} u(t).
\end{align*}
\]

Use Matlab to plot the impulse response of this system (relaxed initial conditions). Now suppose you are given the 1-dimensional system described by the state space equations

\[
\begin{align*}
\dot{x}(t) &= -2x(t) + u(t) \\
y(t) &= x(t).
\end{align*}
\]

Again, use Matlab to plot the impulse response of this system (relaxed initial conditions). Compare your results to the 2-dimensional system. What is going on? Explain analytically.