Solution to Problem #1

- Recall that

$$\det(Q) = \sum_{\sigma \in \mathcal{P}} \operatorname{sgn}(\sigma)\, Q_{1,\sigma_1} \cdot Q_{2,\sigma_2} \cdots Q_{n,\sigma_n}$$

since

$$\alpha Q = \begin{bmatrix} \alpha Q_{11} & \alpha Q_{12} & \cdots & \alpha Q_{1n} \\ \alpha Q_{21} & \alpha Q_{22} & \cdots & \alpha Q_{2n} \\ \vdots & \vdots & & \\ \alpha Q_{n1} & \alpha Q_{n2} & & \alpha Q_{nn} \end{bmatrix}$$

then $$\det(\alpha Q) = \sum_{\sigma \in \mathcal{P}} \operatorname{sgn}(\sigma)(\alpha Q_{1,\sigma_1}) \cdot (\alpha Q_{2,\sigma_2}) \cdots (\alpha Q_{n,\sigma_n})$$

$$= \alpha^n \sum_{\sigma \in \mathcal{P}} \operatorname{sgn}(\sigma)\, Q_{1,\sigma_1} \cdot Q_{2,\sigma_2} \cdots Q_{n,\sigma_n}$$

$$= \alpha^n \det(Q)$$

- If $J = \operatorname{adj}(\mathbf{Q})$, recall that

$$J_{ij} = (-1)^{i+j} \det(M_{ji}) \quad \text{where}$$

M_{ji} is an $(n-1) \times (n-1)$ dimensional matrix formed by deleting row j and column i of $\mathbf{Q}$.

Now, let $J' = \operatorname{adj}(\alpha Q)$, and

$$J'_{ij} = (-1)^{i+j} \det(M'_{ji})$$

It is clear that $M'_{ji} = \alpha M_{ji}$ by the construction of M_{ji}.

Our prior result then implies that

$$\det(M_{ji}') = \alpha^{n-1} \det(M_{ji})$$

since M_{ji} is $(n-1) \times (n-1)$ dimensional.

Hence, $J_{ij}' = (-1)^{i+j} \alpha^{n-1} \det(M_{ji})$

$$= \alpha^{n-1} J_{ij}$$

which implies that $\operatorname{adj}(\alpha Q) = \alpha^{n-1} \operatorname{adj}(Q)$

- Recall that if $\det(Q) \neq 0$,

$$Q^{-1} = \frac{1}{\det(Q)} \operatorname{adj}(Q)$$

Our results imply that

$$(\alpha Q)^{-1} = \frac{1}{\alpha^n \det(Q)} \alpha^{n-1} \operatorname{adj}(Q) \quad (\alpha \neq 0)$$

$$= \frac{1}{\alpha} \cdot \frac{1}{\det(Q)} \operatorname{adj}(Q) \quad (\alpha \neq 0)$$

$$= \frac{1}{\alpha} Q^{-1} \quad (\alpha \neq 0)$$

(as would be expected).

Solution to Problem 2a:

$$v(k) = Px(k) \quad \forall k$$

$$x(k) = P^{-1}v(k) \quad \forall k$$

then $\quad x(k+1) = Ax(k) + Bu(k) \quad$ can be rewritten as

$$P^{-1}v(k+1) = AP^{-1}v(k) + Bu(k)$$

$$\boxed{v(k+1) = PAP^{-1}v(k) + PBu(k)}$$

and similarly, $\boxed{y(k) = CP^{-1}v(k) + Du(k)}$

hence

$$\boxed{\begin{aligned} \bar{A} &= PAP^{-1} \\ \bar{B} &= PB \\ \bar{C} &= CP^{-1} \\ \bar{D} &= D \end{aligned}}$$

Solution to Problem 2b :

Transfer function for (1) is

$$C(sI-A)^{-1}B + D$$

Transfer function for (2) is

$$\bar{C}(sI-\bar{A})^{-1}\bar{B}+\bar{D} \quad \circledast$$

substituting $\bar{C} = CP^{-1}$

$\bar{A} = PAP^{-1}$

$\bar{B} = PB$

$\bar{D} = D$

$$\circledast = CP^{-1}(sI - PAP^{-1})^{-1}PB + D$$

Note that $PP^{-1} = I$ hence

$$\circledast = CP^{-1}(s\,PP^{-1} - PAP^{-1})^{-1}PB + D$$

$$= CP^{-1}(P(sI-A)P^{-1})^{-1}PB + D$$

Note that $(XY)^{-1} = Y^{-1}X^{-1}$ but suppose $X = UV$

then $(XY)^{-1} = Y^{-1}(UV)^{-1} = Y^{-1}V^{-1}U^{-1} = (UVY)^{-1}$

hence

$$\circledast = CP^{-1}P(sI-A)^{-1}P^{-1}PB + D$$

$$= C(sI-A)^{-1}B + D \longrightarrow \text{same as transfer function for (1).}$$

Solution to Problem #3a

From A, we note that

$$\left\{\begin{array}{ll} \dot{x}_1 = x_2 & \Rightarrow s\hat{x}_1(s) = \hat{x}_2(s) \\ \vdots & \\ \dot{x}_{n-1} = x_n & \Rightarrow s\hat{x}_{n-1}(s) = \hat{x}_n(s) \end{array}\right.$$

(we can ignore initial conditions since we are looking at transfer functions here)

This implies that $\hat{x}_n(s) = s^{n-1}\hat{x}_1(s)$. $\leftarrow$ (*)

But, from A, we also see that

$$\dot{x}_n = -a_0 x_1 - \cdots - a_{n-1}x_n + u$$

$$s\hat{x}_n(s) = -a_0\hat{x}_1(s) - \cdots - a_{n-1}\hat{x}_n(s) + \hat{u}(s)$$

plug in (*) ...

$$s^n\hat{x}_1(s) = -a_0\hat{x}_1(s) - \cdots - a_{n-1}s^{n-1}\hat{x}_1(s) + \hat{u}(s)$$

$$\Rightarrow \hat{x}_1(s)\left[s^n + a_{n-1}s^{n-1} + \cdots + a_0\right] = \hat{u}(s)$$

Now, from B, we note that

$$\hat{y}(s) = b_0\hat{x}_1(s) + \cdots + b_{n-1}\hat{x}_n(s)$$

plug in (*) ...

$$\hat{y}(s) = b_0\hat{x}_1(s) + \cdots + b_{n-1}s^{n-1}\hat{x}_1(s)$$

$$= \hat{x}_1(s)\left[b_0 + b_1 s + \cdots + b_{n-1}s^{n-1}\right]$$

and

$$\frac{\hat{y}(s)}{\hat{u}(s)} = \frac{\hat{x}_1(s)\left[b_{n-1}s^{n-1} + \cdots + b_1 s + b_0\right]}{\hat{x}_1(s)\left[s^n + a_{n-1}s^{n-1} + \cdots + a_0\right]} = \hat{g}(s)$$

$\hat{x}(s)$ cancels on numerator and denominator hence A, B, C, D is a realization for $\hat{g}(s)$.

Solution to Problem#3b

Let

$$\hat{h}(s) = \bar{C}(sI-\bar{A})^{-1}\bar{B}+\bar{D} = B^T(sI-A^T)^{-1}C^T + D^T$$

Note that $D=0$, hence we will ignore it in the following.

Note that for any scalar q, $q=q^T$. $\hat{h}(s)$ is a scalar, hence

$$\hat{h}(s) = \left[\hat{h}(s)\right]^T = \left[B^T(sI-A^T)^{-1}C^T\right]^T$$

Recall that $(XY)^T = Y^TX^T$ hence

$$\hat{h}(s) = C\left[(sI-A^T)^{-1}\right]^T B$$

Moreover $(X^{-1})^T = (X^T)^{-1}$ hence

$$\hat{h}(s) = C\left[(sI-A^T)^T\right]^{-1}B$$

$$= C\left[sI^T-(A^T)^T\right]^{-1}B$$

$$= C(sI-A)^{-1}B = \hat{g}(s)$$ as was shown in part a.

Solution to Problem#3c

$$\hat{g}(s) = \frac{s^3}{s^3+2s^2-s+2}$$ → can't exactly use the "formula" from parts a) and b) here because $\deg(N(s)) = \deg(D(s))$

$$\lim_{s\to\infty}\hat{g}(s) = 1$$ → This implies that $D=1$ here. (from problem 1).

We can still use the "template" from part (a) of this problem with some slight modifications:

First let $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & 1 & -2 \end{bmatrix}$ and $B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ from the template.

Then
$$\dot{x}_1 = x_2 \Rightarrow s\hat{x}_1(s) = \hat{x}_2(s)$$
$$\dot{x}_2 = x_3 \Rightarrow s\hat{x}_2(s) = \hat{x}_3(s) \quad (*)$$
$$(**) \quad \dot{x}_3 = -2x_1 + x_2 - 2x_3 + u \Rightarrow s\hat{x}_3(s) = -2\hat{x}_1(s) + \hat{x}_2(s) - 2\hat{x}_3(s) + \hat{u}(s)$$

combine $(*)$ and $(**)$

$$s^3\hat{x}_1(s) = -2\hat{x}_1(s) + s\hat{x}_1(s) - 2s^2\hat{x}_1(s) + \hat{u}(s)$$

$$\Rightarrow \hat{u}(s) = \left[s^3 + 2s^2 - s + 2\right]\hat{x}_1(s)$$

we want $\hat{y}(s) = s^3\hat{x}_1(s)$ so that

cancellation

$$\frac{\hat{y}(s)}{\hat{u}(s)} = \frac{s^3\hat{x}_1(s)}{\left[s^3 + 2s^2 - s + 2\right]\hat{x}_1(s)} = \hat{g}(s)$$

$$\hat{y}(s) = s^3\hat{x}_1(s) \Rightarrow y(t) = \dot{x}_3(t) \quad \text{(zero state response)}$$

but $\dot{x}_3$ is not a state!

However, we know that

$$\dot{x}_3 = -2x_1 + x_2 - 2x_3 + u$$

hence $y = \underbrace{-2x_1 + x_2 - 2x_3}_{\text{these are states}} + \underbrace{u}_{\text{input}}$

hence $y = Cx + Du$

$$C = \begin{bmatrix} -2 & 1 & -2 \end{bmatrix} \qquad D = [1]$$

← (as we expected)

(can confirm with Matlab)

The second realization comes from part(b)

$$\bar{A} = A^T = \begin{bmatrix} 0 & 0 & -2 \\ 1 & 0 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

$$\bar{B} = C^T = \begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}$$

$$\bar{C} = B^T = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$\bar{D} = [1]$$

2nd realization of $\hat{g}(s)$ (confirm with Matlab).

Solution to problem #3d

$$\hat{g}(z) = \frac{z^{-1}}{z^{-2} + 2z^{-1} - 3} = \frac{z}{-3z^2 + 2z + 1}$$

$$= \frac{\left(-\frac{1}{3}\right) z}{z^2 - \frac{2}{3} z - \frac{1}{3}}$$

Use "template" from parts a & b:

Realization #1:

$$A = \begin{bmatrix} 0 & 1 \\ \frac{1}{3} & \frac{2}{3} \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & -\frac{1}{3} \end{bmatrix} \qquad D = [0]$$

Realization #2:

$$\bar{A} = \begin{bmatrix} 0 & 1/3 \\ 1 & 2/3 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ -1/3 \end{bmatrix} \qquad C = \begin{bmatrix} 0 & 1 \end{bmatrix} \qquad D = [0]$$

Solution to Problem #4a

First, compute Jacobian ...

$$f_1 = x_2$$

$$f_2 = \frac{g}{L}\sin x_1 + \frac{f}{LM} x_4 \cos x_1 - \frac{1}{LM} u \cos x_1$$

$$f_3 = x_4$$

$$f_4 = -\frac{f}{M} x_4 + \frac{1}{M} u$$

$$A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_4} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_4} \\ \vdots & & & \vdots \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \cdots & \frac{\partial f_4}{\partial x_4} \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 0 \\ (g/L)\cos x_1 - \frac{f}{LM} x_4 \sin x_1 + \frac{1}{LM} u \sin x_1 & 0 & 0 & \frac{f}{LM}\cos x_1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -\frac{f}{M} \end{bmatrix}$$

$$B = \begin{bmatrix} \frac{\partial f_1}{\partial u} \\ \vdots \\ \frac{\partial f_4}{\partial u} \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{1}{LM}\cos x_1 \\ 0 \\ \frac{1}{M} \end{bmatrix}$$

for C & D, $\quad y = \tan x_1 \Rightarrow g = \tan x_1$

$$C = \begin{bmatrix} \frac{\partial g}{\partial x_1} & \frac{\partial g}{\partial x_2} & \cdots & \frac{\partial g}{\partial x_4} \end{bmatrix}$$

$$= \begin{bmatrix} \sec^2 x_1 & 0 & 0 & 0 \end{bmatrix}$$

$$D = \frac{\partial g}{\partial u} = 0$$

Now plug in solution $x_1 = x_2 = x_3 = x_4 = u = 0$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{g}{L} & 0 & 0 & \frac{f}{LM} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -f/M \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \frac{-1}{LM} \\ 0 \\ \frac{1}{M} \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \quad D = 0$$

Solution to Problem 46

Just plug in solution $x_1 = \pi$, $x_2 = x_3 = x_4 = \quad = 0$

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{g}{L} & 0 & 0 & \frac{-f}{LM} \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -f/M \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ \frac{1}{LM} \\ 0 \\ 1/M \end{bmatrix}$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \quad D = 0$$

Solution to Problem 5

a)
$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 10 \\ 4 & 7 & 13 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}}_{b}$$

$x_1 + 2x_2 + 3x_3 = 0$

$2x_1 + 4x_2 + 6x_3 = 0$

recognize that these equations can both be satisfied with the same x_1, x_2, x_3

Lets put the matrix in echelon form: do Gaussian elimination...

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix}}_{\tilde{b}}$$

(need to swap some rows to truly get echelon form but we can skip that step here)

back substitution...

$\Rightarrow x_3 = 1 \quad \Rightarrow x_2 = -2 \quad \Rightarrow x_1 = 1$

check:

$$\begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 10 \\ 4 & 7 & 13 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \end{bmatrix} \checkmark$$

A has no nullspace, hence $\rightarrow$ Solution is unique. b, in this problem, is in the range of A.

b) We already have echelon form,

$$\underbrace{\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 1 \end{bmatrix}}_{\tilde{A}} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ -1 \\ -2 \\ -1 \end{bmatrix}}_{\tilde{b}}$$

$\leftarrow$ no solution since $0 = -1$ impossible.

$b = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 3 \end{bmatrix}$, in this problem, is <u>not</u> in the range of A.

c) $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 7 \\ 3 & 6 & 10 & 13 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

find echelon form...

$$\begin{array}{c} (1) \\ (2) \\ (3) \end{array} \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ (skipping the row swaps)

clearly $x_4 = 0$ from line (2)

This result and line (3) imply that $x_3 = 0$ as well.

hence (1) $\Rightarrow$ $x_1 + 2x_2 = 0$ or $x_1 = -2x_2$

This results in an infinite number of solutions with basis vector

$$h = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

check $$\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 6 & 7 \\ 3 & 6 & 10 & 13 \end{bmatrix} \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$ ✓

moreover any scalar multiple of h also satisfies

$Ah = 0$.

h is in the nullspace of A and h describes a basis for the nullspace of A.

Solution to problem 6.

- $\boxed{\text{Show that, if the columns of } PQ \text{ are linearly independent, then so are the columns of } Q.}$

We can show this by showing that, if the columns of Q are not linearly independent then the columns of PQ are also not linearly dependent.

Assume $Q = [q_1, \cdots, q_K]$ has columns that are linearly dependent. Then, $\exists \{\alpha_i\}_{i=1}^{K}$ such that

$$\alpha_1 q_1 + \alpha_2 q_2 + \cdots + \alpha_K q_K = 0$$

The columns of PQ may be written as

$$PQ = [Pq_1, Pq_2, \cdots, Pq_K]$$

$$\alpha_1 Pq_1 + \alpha_2 Pq_2 \cdots + \alpha_K Pq_K =$$

$$P(\alpha_1 q_1 + \alpha_2 q_2 + \cdots \alpha_K q_K) = P(0) = 0.$$

Hence the columns of PQ are also linearly dependent. Hence, by contradiction, the boxed claim above is true.

- $\boxed{\text{Show that the converse is not always true}}$

Suppose $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$

Q clearly has linearly independent columns but $PQ = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ does not have linearly independent columns.

This problem illustrates one mechanism for "dilution of rank", e.g. $\text{rank}(AB) \leq \min\{\text{rank}(A), \text{rank}(B)\}$

hence, b and A must be such that the set of vectors (n such $n\times 1$ vectors)

$$\{b, Ab, \ldots, A^{n-1}b\}$$

are all linearly independent in order to meet equation ⊛ for any choice of $x(n)$ and $x(0)$.

Solution to Problem 7

a) The same impulse response means the same transfer function. (We have already got it from Problem 6 of Homework 1)

Then, based on Problem 2 of this homework, we can see that if we define $v[k]=Px[k]$ where P is invertible, then the two systems will have the same transfer function.

I just pick a simple invertible P, like

$$P=\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Then, we can calculate $\bar{A}$, $\bar{B}$, $\bar{C}$, $\bar{D}$ according to our result in 2(a)

$$\bar{A}=PAP^{-1}=\begin{bmatrix} 0 & 3 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

$$\bar{B}=PB=\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$$\bar{C}=CP^{-1}=\begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

$$\bar{D}=D=1$$

Therefore, a different state-space realization of this system could be

$$x[k+1]=\begin{bmatrix}0 & 3 & 1\\ 0 & 0 & 0\\ 0 & 1 & 0\end{bmatrix}x[k]+\begin{bmatrix}1\\0\\0\end{bmatrix}u[k]$$

$$y[k]=\begin{bmatrix}1 & 1 & 1\end{bmatrix}x[k]+u[k]$$

Both of their transfer functions are:

$$\hat{g}(s)=C(sI-A)^{-1}B+D=\frac{1}{s}+1$$

b) Zero input response, $x(k)=A^{k-k_0}x(0)$

$$y(k)=C\,(A^{k-k_0}x(0))$$

$$y(k)=CA^{k}x(0) \quad (\text{LTI system with } k_0=0)$$

Compute $A^0=I_3$

$$A^1=\begin{bmatrix}0 & 1 & 3\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}$$

$$A^2=\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}$$

$$A^3=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}=A^k \text{ for } k\geq 3$$

$$C=\begin{bmatrix}1 & 1 & 1\end{bmatrix}$$

hence

k	$y(k)$ given $x(0)=\begin{bmatrix}1\\0\\0\end{bmatrix}$	$y(k)$, $x(0)=\begin{bmatrix}0\\1\\0\end{bmatrix}$	$y(k)$, $x(0)=\begin{bmatrix}0\\0\\1\end{bmatrix}$
0	$\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=1$	$\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=1$	$\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=1$
1	$\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 3\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=0$	$\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 3\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=1$	$\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 1 & 3\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=4$
2	$\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=0$	$\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\1\\0\end{bmatrix}=0$	$\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 1\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}0\\0\\1\end{bmatrix}=1$
≥ 3	$\begin{bmatrix}1 & 1 & 1\end{bmatrix}\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix}=0$	0	0

hence for

$$x(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad y(0) = 1, \quad y(k) = 0 \quad k \geq 1$$

$$x(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad y(0) = 1, \quad y(1) = 1, \quad y(k) = 0 \quad k \geq 2$$

$$x(0) = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad \underbrace{y(0) = 1, \quad y(1) = 4, \quad y(2) = 1, \quad y(k) = 0 \quad k \geq 3}_{\text{zero input responses.}}$$

c) A general expression for the zero-input response of the system follows directly from linearity ...

k	$y(k)$
0	$\gamma_1 + \gamma_2 + \gamma_3$
1	$\gamma_2 + 4\gamma_3$
2	γ_3
≥ 3	0

Since

$$y(k) = GA^k x(0) = GA^k \left\{ \gamma_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \gamma_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \gamma_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$$

$$= \gamma_1 \underbrace{GA^k \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\text{we computed this in part a}} + \gamma_2 \underbrace{GA^k \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{\text{computed in part a}} + \gamma_3 \underbrace{GA^k \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}}_{\text{computed in part a.}}$$

[END OF SOLUTION #2]