# ECE504 Homework Assignment Number 3
# Due by 8:50pm on 14-Oct-2008

Tips: Make sure your reasoning and work are clear to receive full credit for each problem.

1. 3 pts. Chen problem 3.11.
2. 3 pts. Chen problem 3.16.
3. 4 pts. In each of the following cases, find an input sequence $\{u(k)\}_{k=k_0}^{K-1}$ that drives the specified discrete time system from the given initial state $\boldsymbol{x}(k_0)$ to the given final state $\boldsymbol{x}(K)$. If such an input sequence does not exist, explain why. If an input sequence does exist, determine if it is unique. Justify your answers.

   (a) The discrete time system

   $$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$$

   with $k_0 = 0$, $K = 3$, $\boldsymbol{x}(k_0) = [1, -1, -1]^\top$, and $\boldsymbol{x}(K) = [-5, -3, 1]^\top$.

   (b) The discrete time system

   $$\boldsymbol{A} = \begin{bmatrix} 1 & 2 & 1 \\ 1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

   with $k_0 = 0$, $K = 3$, $\boldsymbol{x}(k_0) = [-1, 2, -4]^\top$, and $\boldsymbol{x}(K) = [2, -6, 1]^\top$.

   (c) The discrete time system

   $$\boldsymbol{A}(k) = \begin{bmatrix} \cos(\pi k) & 1/2 \\ 0 & \sin((\pi/2)k) \end{bmatrix} \quad \boldsymbol{B} = \begin{bmatrix} k \\ 1 \end{bmatrix}$$

   with $k_0 = 2$, $K = 4$, $\boldsymbol{x}(k_0) = [1, 1]^\top$, and $\boldsymbol{x}(K) = [41, -19]^\top$.

4. 3 pts. Solve the first-order, scalar, linear differential equation

   $$\dot{x}(t) = (t-1)(x(t)+1);\ x(t_0) = 0.$$

   Using material that we have covered in ECE504, you should be able to find a closed form solution that doesn't have any integrals in it.

5. 3 pts. Suppose that $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t)$, $t \in \mathbb{R}$, and that $\boldsymbol{A} \in \mathbb{R}^{2\times 2}$ is a constant matrix. You do an experiment and find that when $\boldsymbol{x}(0) = [3, 1]^\top$, $\boldsymbol{x}(t) = [3e^{-2t}, e^{-2t}]^\top$ for all $t \in \mathbb{R}$. You do another experiment and find that when $\boldsymbol{x}(0) = [1, 2]^\top$, $\boldsymbol{x}(t) = [e^{-4t}, 2e^{-4t}]^\top$ for all $t \in \mathbb{R}$. Find $\boldsymbol{A}$ along with $\boldsymbol{\Phi}(t,\tau)$ satisfying the STM differential equations

   $$\text{(STM)} \begin{cases} \frac{d}{dt}\boldsymbol{\Phi}(t,\tau) = \boldsymbol{A}\boldsymbol{\Phi}(t,\tau) \\ \boldsymbol{\Phi}(\tau,\tau) = \boldsymbol{I}_2 \end{cases}$$

6. 3 pts. Consider the system

   $$\dot{\boldsymbol{x}}(t) \quad = \quad \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t)$$

where $\boldsymbol{A}(t) \in \mathbb{R}^{n\times n}$ and $\boldsymbol{B}(t) \in \mathbb{R}^{n\times m}$ (e.g. we have $n$ states and $m$ inputs). Given a particular $\boldsymbol{x}(t_0)$, under what conditions does there exist an input vector $\boldsymbol{u}(t)$, defined for all $t \in \mathbb{R}$, such that $\boldsymbol{x}(t) = \boldsymbol{x}(t_0)$ for all $t \in \mathbb{R}$? Under what conditions is the input vector unique? Use your result to find $u(t)$ for the one-input one-state system

$$\dot{x}(t) = x(t) + e^{-t}u(t)$$

such that $x(t) = x(t_0)$ for all $t \in \mathbb{R}$.

7. 4 pts.
   (a) For arbitrary $t_0 \in \mathbb{R}$, find $\boldsymbol{\Phi}(t, t_0)$ satisfying

   $$(\text{STM}) \begin{cases} \frac{d}{dt}\boldsymbol{\Phi}(t, t_0) = \boldsymbol{A}\boldsymbol{\Phi}(t, t_0) \\ \boldsymbol{\Phi}(\tau, t_0) = \boldsymbol{I}_2 \end{cases}$$

   for all $t \in \mathbb{R}$ when

   $$\boldsymbol{A}(t) = \begin{bmatrix} 3t & 0 \\ t & 0 \end{bmatrix}$$

   (b) What is $\boldsymbol{x}(2)$ when $\boldsymbol{x}(3) = [1, 1]^\top$ and $\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t)$?

8. 3 pts. For each matrix $\boldsymbol{A}$ below, find its characteristic polynomial, its eigenvalues, their algebraic multiplicities, bases for all of the eigenspaces, and the eigenvalues' geometric multiplicities. Comment on diagonalizability.
   (a)

   $$\boldsymbol{A} = \begin{bmatrix} 1 & 4 & 10 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

   (b)

   $$\boldsymbol{A} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

   (c) $\boldsymbol{A} = \boldsymbol{I}_n$

9. 4 pts. Consider the system

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}u(t) \\ y(k) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}u(t) \end{aligned}$$

where $\boldsymbol{B} = \boldsymbol{0}$, $\boldsymbol{D} = \boldsymbol{0}$ and

$$\boldsymbol{A} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & -1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } \boldsymbol{C} = \begin{bmatrix} 1 & 1 & 1 \end{bmatrix}$$

   (a) If possible, select $\boldsymbol{x}(0)$ in such a manner so that $y(t) = 3e^{-t}$ for all $t \in \mathbb{R}$. If you find an answer, is it unique?
   (b) Suppose now that

   $$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \boldsymbol{C} = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

   with $\boldsymbol{B} = \boldsymbol{0}$ and $\boldsymbol{D} = \boldsymbol{0}$. If possible, select $\boldsymbol{x}(0)$ in such a manner so that $y(t) = t/2$ for all $t \in \mathbb{R}$. If you find an answer, is it unique? Hint: This matrix is not diagonalizable so you can't just use the eigenvector/eigenvalue method to find $e^{\boldsymbol{A}t}$. You already know another method - use it!