Solution to Problem 1 (Chen 5.4)

$$\hat{g}(s) = \frac{e^{-2s}}{s+1} \quad \text{, BIBO stable?}$$

$\hat{g}(s)$ is not a rational transfer function — need to look at impulse response

$$g(t) = \mathcal{L}^{-1}[\hat{g}(s)] = \begin{cases} e^{-(t-2)} & \text{for } t \geq 2 \\ 0 & \text{for } t < 2 \end{cases}$$

$$\int_{-\infty}^{\infty} |g(t)|\, dt = \int_{2}^{\infty} e^{-(t-2)} dt \quad \text{, let } \tau = t-2 \text{ , } d\tau = dt$$

$$= \int_{0}^{\infty} e^{-\tau} d\tau = -e^{-\tau}\Big|_{\tau=0}^{\infty} = -(0-1) = 1$$

since $\int_{-\infty}^{\infty} |g(t)|\, dt < \infty$, system is BIBO stable (First Criterion)

Solution to Problem 2 (Chen 5.7)

Is
$$\dot{x} = \begin{bmatrix} -1 & 10 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} -2 \\ 0 \end{bmatrix} u$$

$$y = \begin{bmatrix} -2 & 3 \end{bmatrix} x - 2u$$

BIBO stable?

A is clearly not Hurwitz, can't use Third Critereon.
Our only hope is to get a pole-zero cancellation in the TF.

$$\hat{g}(s) = C(sI-A)^{-1}B + D$$

$$= \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} s+1 & -10 \\ 0 & s-1 \end{bmatrix}^{-1} \begin{bmatrix} -2 \\ 0 \end{bmatrix} - 2$$

$$= \frac{1}{(s+1)(s-1)} \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} s-1 & 10 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} -2 \\ 0 \end{bmatrix} - 2$$

$$= \frac{1}{(s+1)(s-1)} \begin{bmatrix} -2 & 3 \end{bmatrix} \begin{bmatrix} 2-2s \\ 0 \end{bmatrix} - 2 \quad \frac{4s-4}{(s+1)(s-1)} - 2 = \frac{4}{s+1} - 2 = \frac{-2s+2}{s+1}$$

No more pole zero cancellations,
Second critereon says that this system is BIBO stable.

Solution to Problem 3:

$$x(k+1) = \begin{bmatrix} 0.9 & 1 \\ 0 & 1 \end{bmatrix} x(k) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(k)$$

$$y(k) = \begin{bmatrix} c_1 & c_2 \end{bmatrix} x(k)$$

Best approach is probably to find the TF...

$$\hat{g}(z) = C(zI - A)^{-1}B = \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} z-0.9 & -1 \\ 0 & z-1 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$= \frac{1}{(z-0.9)(z-1)} \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} z-1 & 1 \\ 0 & z-0.9 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$$

$$= \frac{1}{(z-0.9)(z-1)} \begin{bmatrix} c_1 & c_2 \end{bmatrix} \begin{bmatrix} b_1(z-1) + b_2 \\ b_2(z-0.9) \end{bmatrix}$$

$$= \frac{c_1 b_1 (z-1) + c_1 b_2 + c_2 b_2 (z-0.9)}{(z-0.9)(z-1)}$$

$$= \frac{(c_1 b_1 + c_2 b_2) z + (c_1 b_2 - c_1 b_1 - 0.9 c_2 b_2)}{(z-0.9)(z-1)}$$

note that we can only have BIBO stability if we can cancel the $(z-1)$ term in the denominator. This can only happen if

$$(\cancel{c_1 b_1} + c_2 b_2) = -c_1 b_2 + \cancel{c_1 b_1} + 0.9 c_2 b_2$$

$$0.1 c_2 \cancel{b_2} = -c_1 \cancel{b_2}$$

Hence our system is BIBO stable if and only if

$\boxed{c_2 = -10 c_1 \text{ for any } b_1 \text{ and } b_2}$ or $\boxed{b_2 = 0 \text{ for any } c_1, c_2, b_1}$

There is no condition on b_1 in general.

Check: $c_1 = 1$, $c_2 = -10$

$$\begin{bmatrix} 1 & -10 \end{bmatrix} \begin{bmatrix} z-1 & 1 \\ 0 & z-0.9 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = b_1(z-1) + b_2 - 10 b_2 (z-0.9)$$

$$= (b_1 - 10 b_2) z - (b_1 - b_2 - 9 b_2)$$

$$= (b_1 - 10 b_2) z - (b_1 - 10 b_2)$$

$$= (b_1 - 10 b_2)(z-1) \leftarrow \text{cancellation with } z-1 \text{ pole.}$$

Solution to problem 4:

$$x(k+1) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(k) + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(k)$$

Reachability =

$$Q_r = [B \;\; AB] = \begin{bmatrix} b_1 & b_2 \\ b_2 & 0 \end{bmatrix}$$

$$\boxed{\begin{aligned} \text{basis for } \{\text{reachable states}\} &= \mathbb{R}^2 \quad \text{if } b_2 \neq 0 \\ &= \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \text{ if } b_2 = 0 \text{ and } b_1 \neq 0 \\ &= \{\emptyset\} \text{ if } b_1 = b_2 = 0 \end{aligned}}$$

Since $\{\text{reachable states}\} \subset \{\text{controllable states}\}$ then we know that

$$\{\text{controllable states}\} = \mathbb{R}^2 \quad \text{if } b_2 \neq 0$$

however, when $b_2 = 0$,

$$x(k) = A^k x(0) + \sum_{\ell=0}^{k-1} A^{k-\ell-1} B u(\ell)$$

and $A^k = 0$ for $k \geq 2$. Hence, we can set $u(\ell) = 0 \;\; \forall \ell$ and we see that any state in $\mathbb{R}^2$ is controllable irrespective of B.

$$\boxed{\text{Hence } \{\text{controllable states}\} = \mathbb{R}^2 \text{ for any } b_1 \text{ and } b_2}$$

$$\boxed{\{\text{reachable states}\} \neq \{\text{controllable states}\} \text{ if and only if } b_2 = 0.}$$

Solution to Problem 5

Since $\bar{x}$ and $\bar{\bar{x}}$ are both reachable states, they are also both controllable states. This implies that there exists an input $w(0), w(1), \dots, w(n-1)$ such that

(1) $x(n) = 0 = A^n \bar{x} + \sum_{\ell=0}^{n-1} A^{n-\ell-1} B w(\ell)$ (drive state from $\bar{x}$ to 0)

and there also exists $v(0), v(1), \dots, v(n-1)$ such that

(2) $x(n) = \bar{\bar{x}} = 0 + \sum_{\ell=0}^{n-1} A^{n-\ell-1} B v(\ell)$ (drive state from 0 to $\bar{\bar{x}}$)

add (1) and (2) together:

$$\bar{\bar{x}} = A^n \bar{x} + \sum_{\ell=0}^{n-1} A^{n-\ell-1} B (w(\ell) + v(\ell))$$

but this is exactly the expression for $x(n)$ given an initial condition $x(0) = \bar{x}$ and an input $u(\ell) = w(\ell) + v(\ell)$.

Hence we have shown the existance of an input that drives the state from $\bar{x}$ to $\bar{\bar{x}}$ in time n.

Solution to Problem 6

Reachable subspace = range (Q_r)

$$Q_r = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

a basis for range (Q_r) is then $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ (one-dimensional)

Unobservable Subspace = nullspace (Q_o)

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & -1 \\ -1 & -1 & 1 \end{bmatrix}$$

looks like it could be full rank but it turns out that $x = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ is in the nullspace of Q_o

a basis for nullspace(Q_o) is then $\left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$ (one dimensional)

The interesting thing is that every reachable state is also an unobservable state.

This system is neither "observable" or "reachable."

Solution to Problem 7:

Suppose $C e^{tA} B = \bar{C} e^{tA} B \quad \forall t \geq 0$

$$\Rightarrow (C-\bar{C}) e^{tA} B = 0 \quad \forall t \geq 0$$

$$\Rightarrow (C-\bar{C}) e^{tA} BB^T e^{tA^T} (C-\bar{C})^T = 0 \quad \forall t \geq 0$$

$$\Rightarrow \int_0^T (C-\bar{C}) e^{tA} BB^T e^{tA^T} (C-\bar{C})^T \, dt = 0 \quad \text{for any } T \geq 0$$

We can factor out the $(C-\bar{C})$ terms since they don't depend on t,

$$\Rightarrow (C-\bar{C}) \underbrace{\left[\int_0^T e^{tA} BB^T e^{tA^T} dt \right]}_{\text{this is the reachability Grammian}} (C-\bar{C})^T = 0$$

Let this equal W.

(recall that reachability & controllability are equivalent concepts in continuous time).

Now, we know that A and B are such that the system is reachable. From Chen thm 6.1 we know that W is nonsingular if A and B are such that the system is reachable.

$$\text{Moreover, } x^T W x = \int_0^T x^T e^{tA} BB^T e^{tA^T} x \, dt$$

$$= \int_0^T \| B^T e^{tA^T} x \|^2 dt \geq 0$$

then W is positive semi-definite for all $T \geq 0$

Since W is non-singular, it can't have any e-values equal to zero hence W must be positive definite here.

But $(C-\bar{C}) W (C-\bar{C})^T = 0$

This is only possible if $C-\bar{C}=0$ thus C must equal $\bar{C}$.

Solution to Problem 8:

$$\dot{x}(t) = -vv^T x(t) + v u(t)$$

Reachable subspace = range (Q_r)

$$Q_r = [B \quad AB \quad \cdots \quad A^{n-1}B]$$
$$= [v \quad -vv^Tv \quad \cdots\cdots \quad (-vv^T)^{n-1} v]$$

note that v^Tv is just a scalar, $v^Tv = \|v\|^2$

now $(-vv^T)^k v = (-1)^k \underbrace{\underbrace{vv^T}\underbrace{vv^T}\cdots\underbrace{vv^T}}_{k \text{ such pairs}} v$

but $v^Tv = \|v\|^2 = \alpha \in \mathbb{R}$

so $(-vv^T)^k = (-1)^k \alpha^k v = \beta_k v$ where $\beta_k \in \mathbb{R}^n$

hence

$$Q_r = [v \quad \beta_1 v \quad \beta_2 v \quad \cdots\cdots \quad \beta_{n-1} v] \quad \text{where } \beta_k \in \mathbb{R}^n, \; k = 1, \ldots, n-1$$

$\Rightarrow$ a basis for range (Q_r) is then $\{v\}$ (one dimensional)

Unobservable subspace = nullspace (Q_o)

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \\ \vdots \\ CA^{n-1} \end{bmatrix} = \begin{bmatrix} v^T \\ v^T(-vv^T) \\ v^T(-vv^T)^2 \\ \vdots \\ v^T(-vv^T)^{n-1} \end{bmatrix}$$

but $v^T(-vv^T)^k = (-1)^k v^T \underbrace{\underbrace{vv^T} \cdots \underbrace{vv^T}}_{k \text{ such pairs}}$

but $v^Tv = \alpha \in \mathbb{R}^n$

$= (-1)^k \alpha^k v^T = \beta_k v^T$

hence $Q_o = \begin{bmatrix} v^T \\ \beta_1 v^T \\ \vdots \\ \beta_{n-1} v^T \end{bmatrix}$ hence if $x \in \text{null}(v^T)$ then $x \in \text{null}(Q_o)$

This is an $n-1$ dimensional subspace of $\mathbb{R}^n$

A basis for nullspace (Q_o) is then $\underbrace{\left\{ \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ -v_1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \\ -v_2 \end{bmatrix} \cdots \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \\ -v_{n-1} \end{bmatrix} \right\}}_{n-1 \text{ vectors, each in } \mathbb{R}^n}$

(end of solution)