ECE504: Lecture 11

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Lecture 11 Major Topics

We are now in Part III of ECE504:

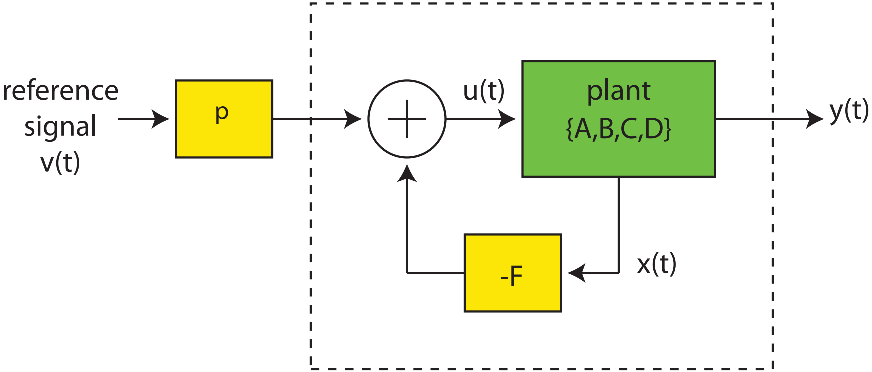
design and control of systems

Today, we will cover:

- State feedback for regulation and tracking
- How to control a system when you can't directly measure the current state.
- State estimators.
- State feedback with estimated states.

You should be reading Chen Chapter 8 now.

Regulation and Tracking



Note the addition of the scalar gain p at the input of the reference signal.

- Regulation: reference signal $v(t) = 0$ for all t.
- Tracking a step input: reference signal $v(t) = c\mathbb{I}(t)$.
- Tracking a general reference signal $v(t)$ is sometimes called "servomechanism theory".

Regulation

- Reference signal $v(t) = 0$ for all t.
- We want the output of the system to also go to zero as $t \to \infty$.
- If the plant is asymptotically stable and $u(t) = 0$, then the output of the system will go to zero as $t \to \infty$.
- What if the plant is not asymptotically stable?
- What if we want the output of the system to go to zero more quickly?

If the system is reachable, we can use state feedback to get any response we like. The zero-input response of the controlled CT-LTI system with state feedback and initial state $\boldsymbol{x}(t_0)$ is

$$\boldsymbol{y}(t) = \boldsymbol{C} \exp\left\{(t - t_0)(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F})\right\} \boldsymbol{x}(t_0)$$

Wonham's eigenvalue assignment theorem says that, as long as the system is reachable, we can find an $\boldsymbol{F}$ such that $\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F}$ has any eigenvalues we want. So we can stabilize a system and get fast regulation.

Tracking a Step Input

Tracking a step input $v(t) = c\mathbb{I}(t)$ is basically the same idea as regulation, except we now want $y(t) \to c$ as $t \to \infty$.

Overall transfer function of the controlled system from $v(t)$ to $y(t)$:

$$\hat{g}(s) = \frac{\hat{y}(s)}{\hat{v}(s)} = \frac{p(b_n s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0)}{s^m + \bar{a}_{m-1}s^{m-1} + \bar{a}_{m-2}s^{m-2} + \ldots \bar{a}_1 s + \bar{a}_0}$$

where the denominator coeffs $\{\bar{a}_i\}$ include the effect of state feedback.

The final value theorem says that, for a step input $v(t) = c\mathbb{I}(t)$,

$$\lim_{t\to\infty} y(t) = c\hat{g}(0) = \frac{cpb_0}{\bar{a}_0}.$$

Hence, as long as $b_0 \neq 0$, we can set $p = \bar{a}_0/b_0$ such that $\lim_{t\to\infty} y(t) = c$.

Wonham's eigenvalue assignment theorem also says that, as long as the system is reachable, we can find an $\boldsymbol{F}$ such that $\boldsymbol{A} - \boldsymbol{BF}$ has any eigenvalues we want. So we can have $y(t) \to c$ quickly as well.

Example: Tracking a Step Input

Suppose

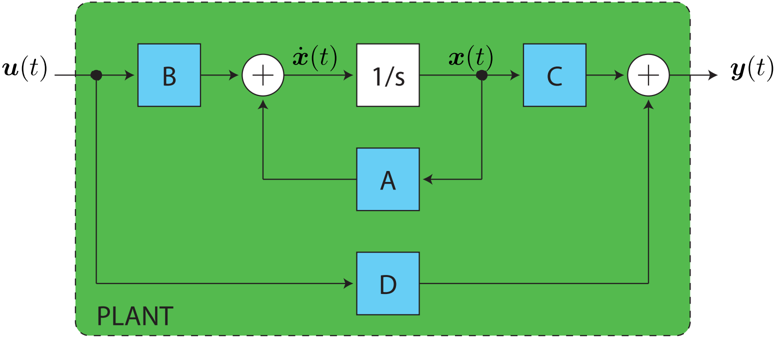
$$A = \begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 1 \end{bmatrix}$$

Suppose we want both e-values of the controlled system to be at -2.

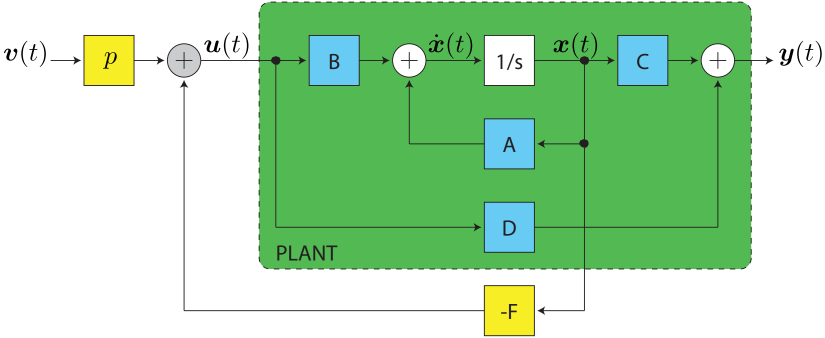
Procedure:

1. Confirm that this is a reachable system.
2. Solve for the state feedback matrix F so that the controlled system has the desired e-values.
3. Compute transfer function of the controlled system.
4. Solve $p = \bar{a}_0 / b_0$.

A New Look at the Uncontrolled System



System With State Feedback (Controlled)



A Dubious Assumption?

A key assumption of state-feedback:

We assume that we can measure the current state of the system.

This may not be possible in many practical systems. What can we do?

One approach:

1. Using the same $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ as the uncontrolled system, build an "auxiliary system" with state $\boldsymbol{w}(t)$.
2. Set $\boldsymbol{w}(0) = \boldsymbol{x}(0)$, then

$$\boldsymbol{w}(t) = \exp\{t\boldsymbol{A}\}\boldsymbol{w}(0) + \int_0^t \exp\{(t-\tau)\boldsymbol{A}\}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau = \boldsymbol{x}(t)$$

Since $\boldsymbol{w}(t) = \boldsymbol{x}(t)$ for all $t \in \mathbb{R}$ (and we built this system, so we have access to its state), we can just use $\boldsymbol{w}(t)$ for state feedback, right?

A More Realistic Approach

One big problem: we can't realistically expect to measure $\boldsymbol{x}(0)$ if we can't measure $\boldsymbol{x}(t)$.

Under the assumption that we know $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ but we don't know $\boldsymbol{x}(t)$ for any t, we can build a state estimator instead:

1. Using the same $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ as the uncontrolled system, build an "auxiliary system" with state $\boldsymbol{w}(t)$.
2. Our auxiliary system observes the output $\boldsymbol{y}(t)$ of the uncontrolled system.
3. We design our auxiliary system so that its state $\boldsymbol{w}(t)$ "tracks" the state of the uncontrolled (or controlled) system $\boldsymbol{x}(t)$ in some sense.
4. We built this system, so we have access to its state.

This auxiliary system is called a "state estimator" or a "state observer".

State Estimator Design (CT-LTI Systems)

If we just guess at $\boldsymbol{w}(0)$, the auxiliary system's state is simply

$$\boldsymbol{w}(t) = \exp\{t\boldsymbol{A}\}\boldsymbol{w}(0) + \int_0^t \exp\{(t-\tau)\boldsymbol{A}\}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau.$$

We would like $\|\boldsymbol{w}(t) - \boldsymbol{x}(t)\| \to 0$, but there is no guarantee that this will happen.

Idea: Use the output $\boldsymbol{y}(t)$ of the uncontrolled system to drive the state of our auxiliary system $\boldsymbol{w}(t)$ to the state of the actual system $\boldsymbol{x}(t)$.

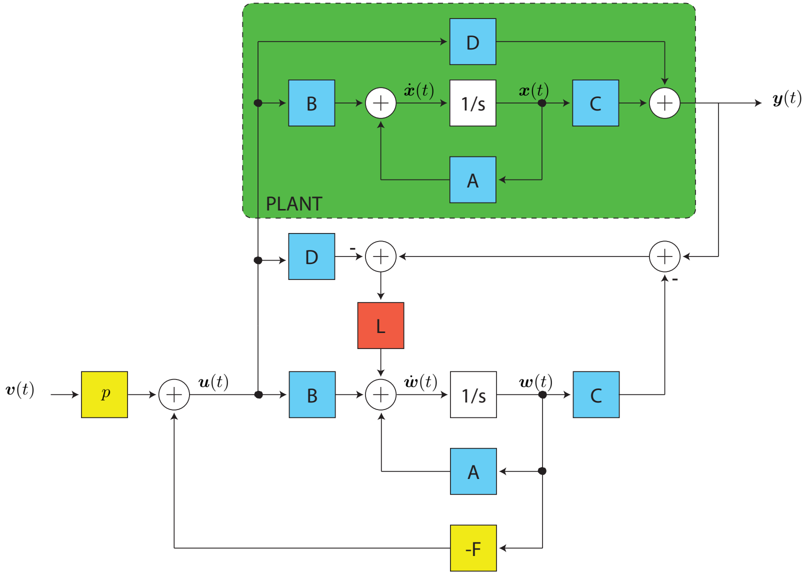
Note that if $\boldsymbol{w}(t) \approx \boldsymbol{x}(t)$, then $\boldsymbol{C}\boldsymbol{w}(t) + \boldsymbol{D}\boldsymbol{u}(t) \approx \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t) = \boldsymbol{y}(t)$.

Let $\boldsymbol{L} \in \mathbb{R}^{n \times q}$ and

$$\boxed{\dot{\boldsymbol{w}}(t) = \boldsymbol{A}\boldsymbol{w}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \underbrace{\boldsymbol{L}\Big(\boldsymbol{y}(t) - \boldsymbol{C}\boldsymbol{w}(t) - \boldsymbol{D}\boldsymbol{u}(t)\Big)}_{\text{correction term}}}$$

It should be clear that the correction term here is small if $\boldsymbol{w}(t) \approx \boldsymbol{x}(t)$.

System With State Feedback Using Estimated State



State Feedback Using Estimated State

To see why this can work, look at the two state update equations

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)\\ \dot{\boldsymbol{w}}(t) &= \boldsymbol{A}\boldsymbol{w}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}\Big(\boldsymbol{y}(t) - \boldsymbol{C}\boldsymbol{w}(t) - \boldsymbol{D}\boldsymbol{u}(t)\Big)\end{aligned}$$

Take the difference to get

$$\frac{d}{dt}(\boldsymbol{x}(t) - \boldsymbol{w}(t)) = \boldsymbol{A}(\boldsymbol{x}(t) - \boldsymbol{w}(t)) - \boldsymbol{L}\Big(\boldsymbol{y}(t) - \boldsymbol{C}\boldsymbol{w}(t) - \boldsymbol{D}\boldsymbol{u}(t)\Big)$$

But $\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)$. So

$$\frac{d}{dt}(\boldsymbol{x}(t) - \boldsymbol{w}(t)) = (\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})(\boldsymbol{x}(t) - \boldsymbol{w}(t))$$

or, more simply

$$\dot{\boldsymbol{z}}(t) = \tilde{\boldsymbol{A}}\boldsymbol{z}(t).$$

Note that this is a CT-LTI system with no input. Under what conditions on $\tilde{\boldsymbol{A}}$ will $\|\boldsymbol{z}(t)\| \to 0$ for any $\boldsymbol{z}(0)$?

State Feedback Using Estimated State

Implications

1. With this approach, we don't need to know $\boldsymbol{x}(t)$ but we do need to know $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ of the uncontrolled system.
2. We can make any guess we want for $\boldsymbol{w}(0)$.
3. Even if our guess $\boldsymbol{w}(0)$ is nowhere near $\boldsymbol{x}(0)$, if $\boldsymbol{A} - \boldsymbol{LC}$ is Hurwitz then $\boldsymbol{w}(t)$ will track $\boldsymbol{x}(t)$ closely after an initial transient.
4. The duration of this transient depends on the real part of the eigenvalues of $\boldsymbol{A} - \boldsymbol{LC}$.

Theorem

Given an uncontrolled CT-LTI system with state-space representation $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}\}$, if $\boldsymbol{A}$ and $\boldsymbol{C}$ are such that the system is observable, then we can place the eigenvalues of $\tilde{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{LC}$ arbitrarily.

The proof is based on Wonham's eigenvalue placement theorem and standard duality arguments between observability and reachability.

Putting it All Together (1 of 2)

Uncontrolled system:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{u}(t)\end{aligned}$$

This system may be unstable or have undesired dynamics. We can often change this behavior with state feedback at the input of the system, i.e. $\boldsymbol{u}(t) = -\boldsymbol{F}\boldsymbol{x}(t) + \boldsymbol{v}(t)$, to get

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F})\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{v}(t) \\ \boldsymbol{y}(t) &= (\boldsymbol{C} - \boldsymbol{D}\boldsymbol{F})\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{v}(t)\end{aligned}$$

- $\boldsymbol{F}$ depends on $\boldsymbol{A}$, $\boldsymbol{B}$, and the desired eigenvalues of the system.
- If the system is reachable, then the system eigenvalues (the eigenvalues of $(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F})$) can be placed anywhere.
- Implementation requires perfect measurements of the state (may be unrealistic).

Putting it All Together (2 of 2)

If we can't directly measure the state of the uncontrolled system, we can build an auxiliary system connected to the output of the actual system

$$\dot{\boldsymbol{w}}(t) = \boldsymbol{A}\boldsymbol{w}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}\Big(\boldsymbol{y}(t) - \boldsymbol{C}\boldsymbol{w}(t) - \boldsymbol{D}\boldsymbol{u}(t)\Big)$$

and do state feedback using the estimated state
i.e. $\boldsymbol{u}(t) = -\boldsymbol{F}\boldsymbol{w}(t) + \boldsymbol{v}(t)$, to get

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{A}\boldsymbol{x}(t) - \boldsymbol{B}\boldsymbol{F}\boldsymbol{w}(t) + \boldsymbol{B}\boldsymbol{v}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}\boldsymbol{x}(t) - \boldsymbol{D}\boldsymbol{F}\boldsymbol{w}(t) + \boldsymbol{D}\boldsymbol{v}(t) \end{aligned}$$

- $\boldsymbol{F}$ depends on $\boldsymbol{A}$, $\boldsymbol{B}$, and the desired eigenvalues of the system.
- $\boldsymbol{L}$ depends on $\boldsymbol{A}$, $\boldsymbol{C}$, and the desired eigenvalues of the estimator.
- If the system is reachable, then the system eigenvalues (the eigenvalues of $(\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F})$) can be placed anywhere.
- If the system is observable, then the state estimator eigenvalues (the eigenvalues of $(\boldsymbol{A} - \boldsymbol{L}\boldsymbol{C})$) can be placed anywhere.

Transfer Function of Controlled System with Actual State

Controlled system with state feedback using the actual state:

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F})\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{v}(t) \\ \boldsymbol{y}(t) &= (\boldsymbol{C} - \boldsymbol{D}\boldsymbol{F})\boldsymbol{x}(t) + \boldsymbol{D}\boldsymbol{v}(t)\end{aligned}$$

The overall transfer function of the controlled system from $v(t)$ to $y(t)$ can easily be calculated as

$$\hat{\boldsymbol{g}}(s) = \frac{\hat{\boldsymbol{y}}(s)}{\hat{\boldsymbol{v}}(s)} = (\boldsymbol{C} - \boldsymbol{D}\boldsymbol{F})(s\boldsymbol{I}_n - (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F}))^{-1}\boldsymbol{B} + \boldsymbol{D}$$

Transfer Function of Controlled System with Estim. State

Controlled system with state feedback using the estimated state:

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) - \boldsymbol{B}\boldsymbol{F}\boldsymbol{w}(t) + \boldsymbol{B}\boldsymbol{v}(t) \quad (1)$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{x}(t) - \boldsymbol{D}\boldsymbol{F}\boldsymbol{w}(t) + \boldsymbol{D}\boldsymbol{v}(t) \quad (2)$$

How can we compute the transfer function in this case?

From the figure on slide 12, we know that

$$\dot{\boldsymbol{w}}(t) = \boldsymbol{A}\boldsymbol{w}(t) + \boldsymbol{B}\boldsymbol{u}(t) + \boldsymbol{L}(\boldsymbol{C}\boldsymbol{x}(t) - \boldsymbol{C}\boldsymbol{w}(t)).$$

Plug in $\boldsymbol{u}(t) = -\boldsymbol{F}\boldsymbol{w}(t) + \boldsymbol{v}(t)$ to write

$$\dot{\boldsymbol{w}}(t) = \boldsymbol{L}\boldsymbol{C}\boldsymbol{x}(t) + (\boldsymbol{A} - \boldsymbol{B}\boldsymbol{F} - \boldsymbol{L}\boldsymbol{C})\boldsymbol{w}(t) + \boldsymbol{B}\boldsymbol{v}(t). \quad (3)$$

Next, define the "super state"

$$\boldsymbol{q}(t) = \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{w}(t) \end{bmatrix}.$$

Write the SS description of our controlled system using the "super state"...

State Feedback with Estimated State Examples

Conclusions

1. This concludes Chapter 8.
2. All of our analysis was for CT-LTI systems, but these ideas also apply directly to DT-LTI systems.
3. Qualitative properties of systems are very important in control:
    - Reachability
    - Observability
4. Lower dimensional problems ($n = 2$ or $n = 3$ with $p = q = 1$) can usually be solved by hand.
5. Higher dimensional problems usually require Matlab.
6. Lots more interesting control topics:
    - Reduced-dimensional state estimators (see Chen section 8.4.1)
    - Optimal control (find an input to minimize a cost function, e.g. energy)
    - ...