ECE504: Lecture 12

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Optimal Control: Problem Setup

We assume CT-LTI systems here.

Optimal control is concerned with the problem of finding an input function $\boldsymbol{u}(t)$ for all $t \geq 0$ so that we minimize a "cost function" of the form

$$J(\boldsymbol{u}(t)) = \int_0^\infty \left(\boldsymbol{z}^\top(t)\boldsymbol{Q}\boldsymbol{z}(t) + \boldsymbol{u}^\top(t)\boldsymbol{R}\boldsymbol{u}(t) \right) dt$$

where

- $\boldsymbol{z}(t) = \boldsymbol{M}\boldsymbol{x}(t)$ for some $\boldsymbol{M} \in \mathbb{R}^{\ell \times n}$.
- $\boldsymbol{Q} \in \mathbb{R}^{\ell \times \ell}$ and $\boldsymbol{R} \in \mathbb{R}^{p \times p}$ are both positive definite

Note that the positive definite assumption implies that $\boldsymbol{z}^\top(t)\boldsymbol{Q}\boldsymbol{z}(t) \geq 0$ and $\boldsymbol{u}^\top(t)\boldsymbol{R}\boldsymbol{u}(t) \geq 0$ for all t, with equality if and only if $\boldsymbol{z}(t) = 0$ and/or $\boldsymbol{u}(t) = 0$. Hence, the cost-function must be non-negative.

This cost function is called the "Linear Quadratic Regulator" (LQR).

Physical Interpretation of LQR

$$J(\boldsymbol{u}(t)) \;=\; \int_0^\infty \left(\boldsymbol{z}^\top(t)\boldsymbol{Q}\boldsymbol{z}(t) + \boldsymbol{u}^\top(t)\boldsymbol{R}\boldsymbol{u}(t)\right) dt$$

Physically:

- $\boldsymbol{z}^\top(t)\boldsymbol{Q}\boldsymbol{z}(t)$ is weighted measure of the power of the state vector
- $\boldsymbol{u}^\top(t)\boldsymbol{R}\boldsymbol{u}(t)$ is a weighted measure of the power of the input

The cost function is a measure of total energy.

Minimizing $J(\boldsymbol{u}(t))$ means that we want to drive the state to zero as quickly as possible, but we don't want to use too much input power to make it happen. The matrices $\boldsymbol{Q}$ and $\boldsymbol{R}$ specify the tradeoff.

Some Extreme Cases

Since $\boldsymbol{z}(t) = \boldsymbol{M}\boldsymbol{x}(t)$, we can write

$$J(\boldsymbol{u}(t)) = \int_0^\infty \left(\boldsymbol{x}^\top(t) \underbrace{\boldsymbol{M}^\top \boldsymbol{Q} \boldsymbol{M}}_{\boldsymbol{W}} \boldsymbol{x}(t) + \boldsymbol{u}^\top(t) \boldsymbol{R} \boldsymbol{u}(t) \right) dt$$

where $\boldsymbol{M}^\top \boldsymbol{Q} \boldsymbol{M} = \boldsymbol{W}$ is positive semi-definite.

Some extreme cases:

- If $\boldsymbol{W}$ is relatively small, then there is little cost for input power, and we know that (assuming the system is controllable) we can drive the state to zero very quickly. Hence control will be fast.
- What happens if $\boldsymbol{W}$ is relatively small? The cost function is dominated by input power. Hence control will be slow.

LQR Analysis

We assume that the CT-LTI system is such that $(\boldsymbol{A}, \boldsymbol{B})$ is reachable.

Theorem (Willems, IEEE Trans. Automatic Control 1971)

The input that minimizes $J(\boldsymbol{u}(t))$ *is given as* $\boldsymbol{u}(t) = -\boldsymbol{F}\boldsymbol{x}(t)$ *where*

$$\boldsymbol{F} = \boldsymbol{R}^{-1}\boldsymbol{B}^\top\boldsymbol{P}$$

and $\boldsymbol{P}$ *is the unique positive definite solution to the algebraic Riccati equation*

$$\boldsymbol{A}^\top\boldsymbol{P} + \boldsymbol{P}\boldsymbol{A} - \boldsymbol{P}\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top\boldsymbol{P} + \boldsymbol{W} = 0$$

Interesting result: State feedback is optimal.

Bad news: The algebraic Riccati equation is not easy to solve in most cases. It is nonlinear in $\boldsymbol{P}$ and may have more than one solution. The positive definite solution, however, is unique.

Example

Suppose

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \qquad \boldsymbol{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \boldsymbol{C} = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad \boldsymbol{D} = 0$$

and $\boldsymbol{W} = \boldsymbol{C}^\top \boldsymbol{C}$ and $\boldsymbol{R} = \rho > 0$. The cost function then becomes

$$J(u(t)) = \int_0^\infty \left(y^2(t) + \rho u^2(t)\right)\, dt$$

This simple example captures the essence of the problem where ρ specifies the tradeoff between state power and input power.

Now set up the algebraic Riccati equation and solve for $\boldsymbol{P}$...

Other Solution Techniques for the Algebraic Riccati Eqn.

1. Matlab functions `care` and `dare` (CT-LTI and DT-LTI, respectively)
2. Hamiltonian matrix (can be used when $(\boldsymbol{A}, \boldsymbol{B})$ are reachable and $(\boldsymbol{A}, \boldsymbol{Q}^{1/2}\boldsymbol{M}$ are observable). Basic idea:
   2.1 Form Hamiltonian matrix

$$\boldsymbol{H} = \begin{bmatrix} \boldsymbol{A} & -\boldsymbol{B}\boldsymbol{R}^{-1}\boldsymbol{B}^\top \\ -\boldsymbol{W} & -\boldsymbol{A}^\top \end{bmatrix} \in \mathbb{R}^{2n \times 2n}$$

   2.2 A fact about the Hamiltonian matrix: $\boldsymbol{H}$ has $2n$ eigenvalues with each eigenvalue mirrored across the imaginary axis.
   2.3 Let $\boldsymbol{V} = [\boldsymbol{v}_1, \dots, \boldsymbol{v}_n] \in \mathbb{C}^{2n \times n}$ be a matrix composed of the eigenvectors corresponding to the eigenvalues with negative real parts.
   2.4 Rewrite

$$\boldsymbol{V} = \begin{bmatrix} \boldsymbol{V}_1 \\ \boldsymbol{V}_2 \end{bmatrix}$$

   2.5 Then $\boldsymbol{P} = \boldsymbol{V}_2 \boldsymbol{V}_1^{-1}$ is the solution to the algebraic Riccati equation.