ECE504: Lecture 1

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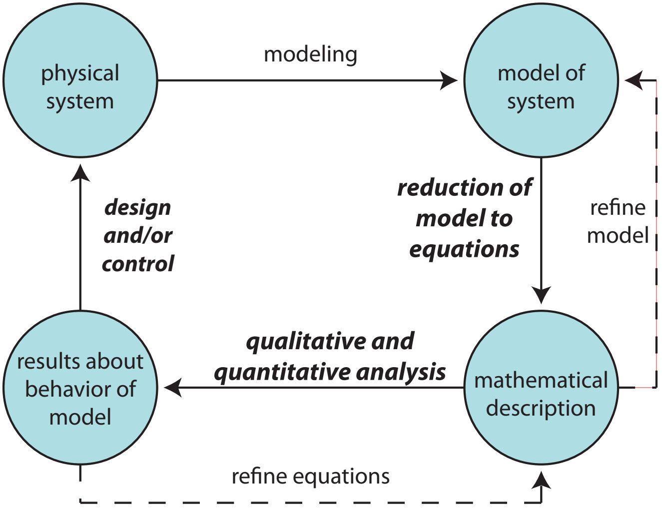
Lecture 1 Major Topics

- Syllabus.
- Course introduction.
- Some examples of simple systems.
- Some qualitative properties of systems.
- Notation.

ECE504: The Big Picture



ECE504: The Big Picture

- **Part I**: Mathematical description of systems (model $\rightarrow$ mathematical description).
- **Part II**: Quantitative and qualitative analysis of systems (mathematical description $\rightarrow$ results about behavior of system).
- **Part III**: Design/modication/control of systems to meet performance criteria (results about behavior of system $\rightarrow$ physical system)

System Modeling

A physical system may have many different models, depending the questions you are asking and the operating conditions of the system.

- Objects moving slowly: Newtonian physics.
- Objects moving quickly: Relativity (Einstein).
- Simple resistor model: $v = iR$.
- More complicated resistor models including stray inductance, temperature effects, wattage limitations, etc. Example: $v = i(R + (T - 273)R/100)$ where T is the temperature in Kelvin.

"Selecting a model that is close enough to a physical system and yet simple enough to be studied analytically is the most difficult and important problem in system design," Chen Chapter 1.

System modeling, however, is application/discipline specific. In most problems in ECE504, we assume we've been given the model..

Part I: Mathematical Descriptions of Systems

In ECE504, when we refer to the "system", we usually mean the *model of the system*, not the actual *physical system*.

A model of a system may also have many different mathematical descriptions. What are some mathematical descriptions of systems that you have already seen?

In ECE504, we are interested in understanding the different types of mathematical descriptions available to us, their advantages and limitations, analysis techniques, and applications to system design and control.

Common Mathematical Descriptions of Dynamic Systems

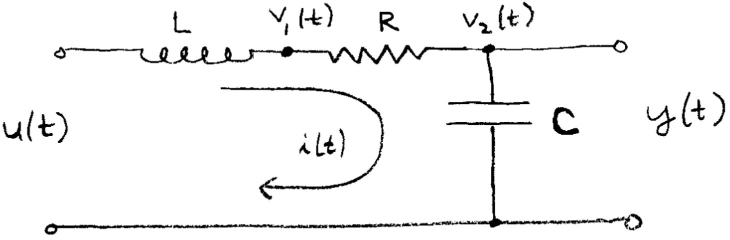
- Input-output differential/difference equation
- Transfer function (Laplace/Z)
- Frequency response (Fourier series, Fourier transform, DFT, DTFT, ...)
- Impulse/step/ramp response
- State-space

These descriptions are related but not equivalent, in general.

Part I: Mathematical Descriptions of Systems

- Review mathematical descriptions of dynamic systems.
- Develop the state-space description.
  - Understand its advantages and limitations.
  - Learn linear algebraic tools for analyzing state-space descriptions.
- Derive relationships between descriptions.

Example: Simple Linear Circuit



- $u(t)$, $-\infty < t < \infty$, is the input.
- $y(t)$, $-\infty < t < \infty$ is the output.
- We want to find the input-output description of this system, i.e. $y(t)$ and its derivatives in terms of $u(t)$ and its derivatives.

Let's derive an input/output differential equation mathematical description of this system...

Example: Simple Linear Circuit Input-Output Description

$$\frac{d^2y}{dt^2} + \frac{R}{L}\frac{dy}{dt} + \frac{1}{LC}y = \frac{1}{LC}u$$

Typical questions:

- For what values of L, C, and R is this system "stable"?
- If $u(t) = a\sin(\omega t + \phi)$, what is $y(t)$?
- What frequencies are attenuated by more than 3dB?
- What is the response of the system to
  - an impulse?
  - a step?
  - a ramp?
- Etc.

This system is simple enough that we could actually answer all of these questions using tools that you learned in undergraduate signals/systems.

Example: Simple Linear Circuit State-Space Description

Let's derive a transfer function mathematical description of this system...

Let's derive a state-space mathematical description of this system...

Are the DE/TF/SS descriptions equivalent?

Example: Sharks and Sardines

A Mathematical Theory of the Struggle for Life, Vito Volterra, 1924

Volterra used fishing statistics from 1914 to 1923 to develop a model and a mathematical description of sharks and sardines in Italian fishing waters.

Notation:

- $x(t)$ is the number of food fish, known collectively as "sardines"
- $y(t)$ is the number of predator fish, known collectively as "sharks"

Basic assumptions in Volterra's model:

- If there were no sharks, the number of sardines $x(t)$ would grow exponentially.
- If there were no sardines, the number of sharks $y(t)$ would decay exponentially.
- The product $x(t)y(t)$ is proportional to the number of meetings between sharks and sardines. These meetings are good for the sharks ($\uparrow$ population) and bad for the sardines ($\downarrow$ population).

Example: Sharks and Sardines

Mathematical description:

$$\begin{aligned}\dot{x}(t) &= ax(t) - cx(t)y(t)\\ \dot{y}(t) &= -by(t) + fx(t)y(t)\end{aligned}$$

where $a > 0$, $b > 0$, $c > 0$, and $f > 0$ are constants that can be selected to fit the model to the fishing statistics. Does this mathematical description satisfy all of Volterra's assumptions?

Note that there is no "input" here. There can't be an input-output description for this system. What can we do?

Analysis...

Example: Saving Money

- You receive a paycheck every two weeks. Let k be an integer index of pay periods. The paycheck amount in pay-period k is denoted as $u[k]$.
- You put a fraction $0 \le \alpha \le 1$ of your pay into your savings account and the rest into a money market account.
- The savings account annual interest rate is denoted as r_s and is compounded weekly.
- The money market account annual interest rate is denoted as r_m and is compounded daily.
- Let $x_s[k]$ and $x_m[k]$ be the dollar values of the savings and money market accounts, respectively, prior to receipt of a paycheck in pay period k.

$$x_s[k+1] = \alpha u[k] + \left(1 + \frac{r_s}{52}\right)^2 x_s[k]$$

$$x_m[k+1] = (1-\alpha) u[k] + \left(1 + \frac{r_m}{365}\right)^{14} x_m[k]$$

State space description ...

More Examples

See Section 2.5 of Chen for several more illustrative examples (including mechanical examples).

Continuous-Time and Discrete-Time Systems

Definition (Continuous-Time System)

A continuous time system accepts continuous-time signals at its input $\boldsymbol{u}(t)$ and generates continuous-time signals at its output $\boldsymbol{y}(t)$ where $-\infty < t < \infty$ can take on any value on the real line.

Definition (Discrete-Time System)

A discrete time system accepts discrete-time signals at its input $\boldsymbol{u}[k]$ and generates discrete-time signals at its output $\boldsymbol{y}[k]$ where $-\infty < k < \infty$ can take on any integer value.

We typically denote the real line as $\mathbb{R} = (-\infty, \infty)$ and the set of all integers as $\mathbb{Z} = \{\dots, -1, 0, 1, \dots\}$.

Either type of system can have multiple inputs and/or multiple outputs.

Qualitative Properties of Systems: Memory

Definition (Memoryless System)

A system in which the output at time t (or k) only depends on the input at time t (or k).

Examples?

Definition (Dynamic System)

A system in which the output at time t (or k) may depend on past, present, and future inputs.

Examples?

Qualitative Properties of Systems: Causality

Definition (Causal System)

A system in which the present output depends only on the past and present inputs, but not future inputs.

Examples?

Remarks

1. All memoryless systems are causal.
2. Aren't all practical systems causal?

Qualitative Properties of Systems: State

Definition (State of a system at time t_0)

The state $\boldsymbol{x}(t_0)$ of a system at time t_0 is the information at t_0 that, together with the input $\boldsymbol{u}(t)$ for all $t \geq t_0$, uniquely determines the output $\boldsymbol{y}(t)$ for all $t \geq t_0$.

$$\left.\begin{array}{l}\boldsymbol{x}(t_0) \\ \boldsymbol{u}(t),\ t \geq t_0\end{array}\right\} \rightarrow \boldsymbol{y}(t),\ t \geq t_0$$

For discrete time systems, replace t_0 with k_0 and replace t with k.

Qualitative Properties of Systems: State

Remarks:

- Intuitively, the state completely summarizes the past inputs $\boldsymbol{u}(t)$ for $t \in (-\infty, t_0)$ on present and future outputs.
- In our circuit example, can we just use $\boldsymbol{x}(t) = v_1(t)$ as the state? What about $\boldsymbol{x}(t) = v_2(t)$? What about $\boldsymbol{x}(t) = [v_1(t), v_2(t)]^\top$?
- The state is also related to the "initial conditions" of the system.
- Note that the choice of state is not unique, in general.

Qualitative Properties of Systems: Lumpedness

Definition (Lumped System)

A system where the number of state variables is finite (but greater than zero).

A system with no state variables is not a lumped system. It is __________.

Examples of lumped systems?

Remarks

1. The opposite of "lumped" is "distributed".
2. Suppose $y(t) = u^2(t)$. Is this memoryless, lumped, or distributed?
3. Suppose $y(t) = u^2(t-1)$. Is this memoryless, lumped, or distributed?
4. Suppose $y[k] = u^2[k-1]$. Is this memoryless, lumped, or distributed?

Qualitative Properties of Systems: Linearity

$$\left.\begin{array}{l} \boldsymbol{x}_1(t_0) \\ \boldsymbol{u}_1(t),\ t \ge t_0 \end{array}\right\} \to \boldsymbol{y}_1(t),\ t \ge t_0 \qquad \left.\begin{array}{l} \boldsymbol{x}_2(t_0) \\ \boldsymbol{u}_2(t),\ t \ge t_0 \end{array}\right\} \to \boldsymbol{y}_2(t),\ t \ge t_0$$

Definition (Linear System)

For any t_0, $\boldsymbol{x}_1(t_0)$, $\boldsymbol{x}_2(t_0)$, $\boldsymbol{u}_1(t)$, and $\boldsymbol{u}_2(t)$ $t \ge t_0$, a system is linear if it is *additive*

$$\left.\begin{array}{l} \boldsymbol{x}_1(t_0) + \boldsymbol{x}_2(t_0) \\ \boldsymbol{u}_1(t) + \boldsymbol{u}_2(t),\ t \ge t_0 \end{array}\right\} \to \boldsymbol{y}_1(t) + \boldsymbol{y}_2(t),\ t \ge t_0$$

and *homogeneous*

$$\left.\begin{array}{l} \alpha\boldsymbol{x}_1(t_0) \\ \alpha\boldsymbol{u}_1(t),\ t \ge t_0 \end{array}\right\} \to \alpha\boldsymbol{y}_1(t),\ t \ge t_0.$$

Qualitative Properties of Systems: Linearity

When a system is **linear**:

- The total system response can be decomposed into the "zero-input response" and the "zero-state response".
- Zero-input response: Set $\boldsymbol{u}_1(t) \equiv 0$, $\boldsymbol{u}_2(t) = \boldsymbol{u}(t)$ for $t \geq 0$.
- Zero-state response: Set $\boldsymbol{x}_2(t_0) = 0$, $\boldsymbol{x}_1(t_0) = \boldsymbol{x}(t_0)$.
- Then given the state $\boldsymbol{x}_1(t_0) + \boldsymbol{x}_2(t_0) = \boldsymbol{x}(t_0)$ and the input $\boldsymbol{u}_1(t) + \boldsymbol{u}_2(t) = \boldsymbol{u}(t)$ for all $t \geq t_0$, we have the uniquely determined output $\boldsymbol{y}_1(t) + \boldsymbol{y}_2(t) = \boldsymbol{y}(t)$ for all $t \geq t_0$.

The idea here is that we can study the effects of state and input **separately** in linear systems.

This is not true, in general, for nonlinear systems.

Qualitative Properties of Systems: Linearity

- The bad news: Nearly all physical systems are nonlinear.
  - Resistance changes as a function of the current passing through a resistor ($T = f(i)$, i.e. temperature effects).
  - Limited power supply voltages cause clipping in linear amplifiers.
  - Overstretched or overcompressed springs.
- The good news: We can examine many interesting nonlinear systems in an operating regime that is well-modeled as linear.
- Example: linear spring model $F = -kx$. This model is a pretty good approximation of reality as long as the spring is not overstreched or overcompressed.
- More good news: Smooth nonlinear systems can often be linearized around a particular operating point to aid in analysis.
- Example: $\sin(x) \approx x$ and $\cos(x) \approx 1$ for small values of x. This approximation is used to analyze the dynamics of lots of different systems with rotation, e.g. a pendulum. See Example 2.8 in Chen.

Qualitative Properties of Systems: Time-Invariance

$$\left.\begin{array}{l} \boldsymbol{x}(t_0) \\ \boldsymbol{u}(t),\ t \geq t_0 \end{array}\right\} \rightarrow \boldsymbol{y}(t),\ t \geq t_0$$

Definition (Time-Invariant System)

For any t_0, $\boldsymbol{x}(t_0)$, $\boldsymbol{u}(t)$ $t \geq t_0$, and T, a system is time invariant if

$$\left.\begin{array}{l} \boldsymbol{x}(t_0 + T) \\ \boldsymbol{u}(t - T),\ t \geq t_0 \end{array}\right\} \rightarrow \boldsymbol{y}(t - T),\ t \geq t_0.$$

- Circuit example: time-invariant or time-varying?
- Sharks and sardines example: time-invariant or time-varying? What if a, b, c, and f were seasonal, e.g. sharks had less appetite in the winter?
- Savings example: time-invariant or time-varying? What if interest rates were variable (as they are in the real world)?

Some Advantages of State-Space Descriptions

1. Explicit description of internal system behavior, not just input/output.
2. Useful for describing both continuous-time and discrete-time systems.
3. Useful for describing both time-varying and time-invarant systems.
4. Useful for describing both linear and non-linear systems.
5. Relatively easy to simulate on computers. Results also tend to be more accurate than direct simulation of input/output equations.
6. Second-order systems can be solved graphically via "phase-plane analysis" (even for nonlinear systems, as we saw).
7. Lots of qualitative analysis techniques available, e.g. stability, controllability, etc.
8. State-space description is more general than "transfer function" description (Laplace/Z/Fourier).
9. Linear state-space systems (the main focus of ECE504) are analyzed primarily with linear algebra, not calculus.

Some Limitations of State-Space Descriptions

- Can't handle distributed systems. State-space can only represent lumped systems.
- Can't handle noncausal systems.

In these cases, you will have to work directly with the input/output differential/difference description, impulse response, or transfer function.

Some Mathematical Notation

- $\mathbb{R}$ denotes the set of real numbers.
- $\mathbb{R}_+$ denotes the set of real non-negative numbers (≥ 0).
- $\mathbb{R}^n$ denotes the set of n dimensional column vectors whose elements are real numbers, e.g. $\boldsymbol{x} \in \mathbb{R}^3$.
- $\mathbb{R}^{n \times m}$ denotes the set of $n \times m$ matrices (n rows, m columns) whose entries are real numbers, e.g. $\boldsymbol{A} \in \mathbb{R}^{4 \times 3}$.
- $\mathbb{C}$ denotes the set of complex numbers.
- $\mathbb{C}^n$ denotes the set of n dimensional column vectors whose elements are complex numbers.
- $\mathbb{C}^{n \times m}$ denotes the set of $n \times m$ matrices (n rows, m columns) whose entries are complex numbers.
- $\mathbb{Z}$ denotes the set of integers $Z = \{0, \pm 1, \pm 2, \cdots\}$.
- $\mathbb{N}$ denotes the set of natural numbers $N = \{0, 1, 2, \cdots\}$.

Some More Mathematical Notation

- Boldface lowercase variables, e.g. $\boldsymbol{x}$, usually represent vectors. Unless otherwise stated, we assume column vectors.
- A row vector is usually indicated by transpose or Hermetian. For example, if

$$\boldsymbol{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in \mathbb{R}^2 \text{ (column vector)}$$

then

$$\boldsymbol{x}^\top = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \in \mathbb{R}^{1\times 2} \text{ (row vector).}$$

- Boldface uppercase variables, e.g. $\boldsymbol{A}$, usually represent matrices.
- The time variable in continuous-time systems is usually t and we use parentheses, e.g. $\boldsymbol{x}(t)$, to explicitly indicate dependence on t.
- The time variable in discrete-time systems is usually k and we use brackets, e.g. $\boldsymbol{x}[k]$, to explicitly indicate dependence on k.