ECE504: Lecture 3

D. Richard Brown III

Worcester Polytechnic Institute

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Lecture 3 Major Topics

We are finishing up Part I of ECE504: **Mathematical description of systems**

model $\rightarrow$ mathematical description

Today:

1. Time varying systems
2. Linearization of smooth nonlinear systems
3. Examples

This concludes Chen Chapter 2. You may also want to begin reading ahead in Chen Chapter 3.

What About Linear Time-Varying (LTV) Systems?

- I/O differential/difference equation representation ok?
- Impulse response representation ok?
- Transfer function representation ok?
- State-space representation ok?

The continuous-time state-space representation becomes

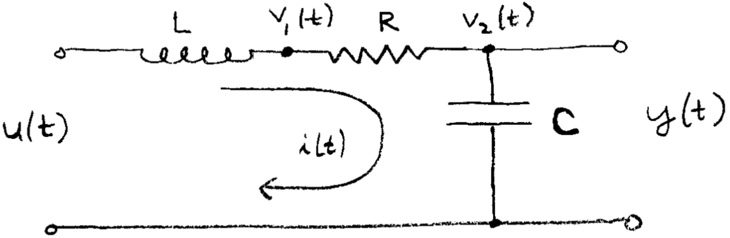
$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t)\\ \boldsymbol{y}(t) &= \boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{D}(t)\boldsymbol{u}(t)\end{aligned}$$

and the discrete-time state-space representation becomes

$$\begin{aligned}\boldsymbol{x}[k+1] &= \boldsymbol{A}[k]\boldsymbol{x}[k] + \boldsymbol{B}[k]\boldsymbol{u}[k]\\ \boldsymbol{y}[k] &= \boldsymbol{C}[k]\boldsymbol{x}[k] + \boldsymbol{D}[k]\boldsymbol{u}[k]\end{aligned}$$

where the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ can change over time.

Example: Linear Time-Varying System



Suppose the resistor is time varying now, i.e. $R = R(t)$.

Same procedure as lecture 1 to derive the I/O differential equation description (just use standard KVL and KCL):

$$\frac{d^2y(t)}{dt^2} + \frac{R(t)}{L}\frac{dy(t)}{dt} + \frac{1}{LC}y(t) = \frac{1}{LC}u(t)$$

Example: Linear Time-Varying System

I/O differential equation description:

$$\frac{d^2y(t)}{dt^2} + \frac{R(t)}{L}\frac{dy(t)}{dt} + \frac{1}{LC}y(t) = \frac{1}{LC}u(t)$$

No transfer function representation. How can we write the continuous-time state-space representation?

$$\begin{aligned}\dot{\boldsymbol{x}}(t) &= \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t) \\ \boldsymbol{y}(t) &= \boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{D}(t)\boldsymbol{u}(t)\end{aligned}$$

What About Nonlinear Systems?

Nonlinear continuous-time state-space description:

$$\begin{aligned} \dot{\boldsymbol{x}}(t) &= \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(t)) \\ \boldsymbol{y}(t) &= \boldsymbol{g}(t, \boldsymbol{x}(t), \boldsymbol{u}(t)) \end{aligned}$$

where

$$\begin{aligned} \dot{x}_1(t) &= f_1(t, x_1(t), \dots, x_n(t), u_1(t), \dots, u_p(t)) \\ \vdots & \quad \vdots \quad \vdots \\ \dot{x}_n(t) &= f_n(t, x_1(t), \dots, x_n(t), u_1(t), \dots, u_p(t)) \end{aligned}$$

and

$$\begin{aligned} y_1(t) &= g_1(t, x_1(t), \dots, x_n(t), u_1(t), \dots, u_p(t)) \\ \vdots & \quad \vdots \quad \vdots \\ y_q(t) &= g_q(t, x_1(t), \dots, x_n(t), u_1(t), \dots, u_p(t)) \end{aligned}$$

A Simple Example of a Nonlinear System

A nonlinear system with one state:

$$\begin{aligned} \dot{x}(t) &= (x(t)-1)^2 + u(t) \\ y(t) &= \tanh(x(t)) \end{aligned}$$

One solution to this nonlinear system is $x(t) \equiv 1$ and $u(t) \equiv 0$. The output in this case is $y(t) \equiv 0$.

What do you think will happen if we apply $x(0) = 0$ and $u(t) \equiv 0$? You can use your intuition here...

Linearization (Step 1a)

As mentioned in Lecture 1, "smoothly nonlinear" systems can often be linearized around a particular operating point, resulting in a standard linear state-space description that can be analyzed with linear algebra.

How can we do this?

Step 1: Find a solution to the state dynamics equation $\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(t, \boldsymbol{x}(t), \boldsymbol{u}(t))$ that holds for all $t \in \mathbb{R}$.

- Often, this is something like $\boldsymbol{x}(t) = \boldsymbol{0}$ and $\boldsymbol{u}(t) = \boldsymbol{0}$ (the system is relaxed with no input).
- There is likely to be more than one possible solution. You should pick the one that represents the nominal operating conditions around which you wish to analyze the behavior of the system.
- Call this nominal solution $\tilde{\boldsymbol{x}}(t)$ and $\tilde{\boldsymbol{u}}(t)$.

Taylor Series Approximation Review

Recall that the Taylor series expansion of a scalar function $f : \mathbb{R} \mapsto \mathbb{R}$ infinitely differentiable in the neighborhood of point x_0 is given as

$$f(x) \;=\; f(x_0) + \frac{f'(x_0)}{1!}(x - x_0) + \frac{f''(x_0)}{2!}(x - x_0)^2 + \dots$$

Consider the first-order Taylor series approximation at the point x_0

$$f(x) \approx f(x_0) + f'(x_0)(x - x_0) = \tilde{f}_1(x)$$

If we let $x = x_0 + \epsilon$, we can rewrite this last expression as

$$f(x_0 + \epsilon) \approx f(x_0) + f'(x_0)\epsilon = \tilde{f}_{x_0}(\epsilon)$$

Since x_0 is fixed, $f(x_0)$ and $f'(x_0)$ are just numbers (not functions of ϵ). Hence, what can you say about $\tilde{f}_{x_0}(\epsilon)$?

Taylor Series Approximation for Vector Functions

The same idea applies to vector functions $\boldsymbol{f} : \mathbb{R}^m \mapsto \mathbb{R}^n$. If the vector function $\boldsymbol{f}$ is differentiable at the point $\boldsymbol{x}_0$, then the first-order Taylor series approximation of $\boldsymbol{f}$ at the point $\boldsymbol{x}_0$ can be written as

$$\boldsymbol{f}(\boldsymbol{x}_0 + \boldsymbol{\epsilon}) \approx \boldsymbol{f}(\boldsymbol{x}_0) + \frac{\partial \boldsymbol{f}(\boldsymbol{x}_0)}{\partial \boldsymbol{x}} \boldsymbol{\epsilon} = \tilde{f}_{\boldsymbol{x}_0}(\boldsymbol{\epsilon})$$

where

$$\frac{\partial \boldsymbol{f}(\boldsymbol{x}_0)}{\partial \boldsymbol{x}} := \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \dots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\text{evaluated at } \boldsymbol{x}=\boldsymbol{x}_0}$$

Note that $\boldsymbol{f}(\boldsymbol{x}_0)$ is just a vector and $\frac{\partial \boldsymbol{f}(\boldsymbol{x}_0)}{\partial \boldsymbol{x}}$ is just a matrix. Hence $\tilde{f}_{\boldsymbol{x}_0}(\boldsymbol{\epsilon}) = \boldsymbol{\gamma} + \boldsymbol{\Gamma}\boldsymbol{\epsilon}$ is linear (affine) in ϵ.

Linearization (Step 1b)

So we now have the nominal solution

$$\begin{aligned}\dot{\tilde{\boldsymbol{x}}}(t) &= \boldsymbol{f}(t, \tilde{\boldsymbol{x}}(t), \tilde{\boldsymbol{u}}(t)) = \boldsymbol{f}(t, \tilde{\boldsymbol{v}}(t)) \\ \tilde{\boldsymbol{y}}(t) &= \boldsymbol{g}(t, \tilde{\boldsymbol{x}}(t), \tilde{\boldsymbol{u}}(t)) = \boldsymbol{g}(t, \tilde{\boldsymbol{v}}(t))\end{aligned}$$

where $\tilde{\boldsymbol{v}}(t) = [\tilde{\boldsymbol{x}}^\top(t), \tilde{\boldsymbol{u}}^\top(t)]^\top$. Our "smooth" assumption on $\boldsymbol{f}$ implies that small changes in the input and initial state lead to small changes in the solution. First-order Taylor series approximation of state dynamics:

$$\begin{aligned}\dot{\tilde{\boldsymbol{x}}}(t) + \dot{\boldsymbol{x}}_\epsilon(t) &\approx \boldsymbol{f}\left(t, \tilde{\boldsymbol{v}}(t) + \boldsymbol{v}_\epsilon(t)\right) \\ &\approx \boldsymbol{f}\left(t, \tilde{\boldsymbol{v}}(t)\right) + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{v}}\right|_{\boldsymbol{v}=\tilde{\boldsymbol{v}}} \boldsymbol{v}_\epsilon(t)\end{aligned}$$

The nominal solution says that $\dot{\tilde{\boldsymbol{x}}}(t) = \boldsymbol{f}(t, \tilde{\boldsymbol{v}}(t))$, hence

$$\dot{\boldsymbol{x}}_\epsilon(t) \approx \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{v}}\right|_{\boldsymbol{v}=\tilde{\boldsymbol{v}}} \boldsymbol{v}_\epsilon(t)$$

Linearization (Step 1b continued)

Recall that

$$\boldsymbol{v}(t) = \begin{bmatrix} \boldsymbol{x}(t) \\ \boldsymbol{u}(t) \end{bmatrix}, \ \tilde{\boldsymbol{v}}(t) = \begin{bmatrix} \tilde{\boldsymbol{x}}(t) \\ \tilde{\boldsymbol{u}}(t) \end{bmatrix}, \text{ and } \boldsymbol{v}_\epsilon(t) = \begin{bmatrix} \boldsymbol{x}_\epsilon(t) \\ \boldsymbol{u}_\epsilon(t) \end{bmatrix}$$

Hence,

$$\left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{v}}\right|_{\boldsymbol{v}=\tilde{\boldsymbol{v}}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_p} \\ \vdots & & \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_p} \end{bmatrix}$$

evaluated at $\boldsymbol{x}(t) = \tilde{\boldsymbol{x}}(t)$ and $\boldsymbol{u}(t) = \tilde{\boldsymbol{u}}(t)$.

Linearization (Step 1b continued)

$$
\begin{aligned}
\left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{v}}\right|_{\boldsymbol{v}=\tilde{\boldsymbol{v}}} \boldsymbol{v}_\epsilon(t) &= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} & \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_p} \\ \vdots & & \vdots & & & \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} & \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_p} \end{bmatrix}_{\boldsymbol{x}(t)=\tilde{\boldsymbol{x}}(t),\boldsymbol{u}(t)=\tilde{\boldsymbol{u}}(t)} \begin{bmatrix} \boldsymbol{x}_\epsilon(t) \\ \boldsymbol{u}_\epsilon(t) \end{bmatrix} \\
&= \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}_{\boldsymbol{x}(t)=\tilde{\boldsymbol{x}}(t),\boldsymbol{u}(t)=\tilde{\boldsymbol{u}}(t)} \boldsymbol{x}_\epsilon(t) \\
&\quad + \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_p} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_p} \end{bmatrix}_{\boldsymbol{x}(t)=\tilde{\boldsymbol{x}}(t),\boldsymbol{u}(t)=\tilde{\boldsymbol{u}}(t)} \boldsymbol{u}_\epsilon(t) \\
&= \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\tilde{\boldsymbol{x}},\,\boldsymbol{u}=\tilde{\boldsymbol{u}}} \boldsymbol{x}_\epsilon(t) + \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_{\boldsymbol{x}=\tilde{\boldsymbol{x}},\,\boldsymbol{u}=\tilde{\boldsymbol{u}}} \boldsymbol{u}_\epsilon(t)
\end{aligned}
$$

Linearization (Step 1c)

Similarly, our "smooth" assumption on $\boldsymbol{g}$ implies that

$$\begin{aligned}\tilde{\boldsymbol{y}}(t) + \boldsymbol{y}_\epsilon(t) &\approx \boldsymbol{g}\left(t, \tilde{\boldsymbol{x}}(t) + \boldsymbol{x}_\epsilon(t), \tilde{\boldsymbol{u}}(t) + \boldsymbol{u}_\epsilon(t)\right) \\ &\approx \boldsymbol{g}\left(t, \tilde{\boldsymbol{x}}(t), \tilde{\boldsymbol{u}}(t)\right) + \left.\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\tilde{\boldsymbol{x}},\, \boldsymbol{u}=\tilde{\boldsymbol{u}}} \boldsymbol{x}_\epsilon(t) + \left.\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}}\right|_{\boldsymbol{x}=\tilde{\boldsymbol{x}},\, \boldsymbol{u}=\tilde{\boldsymbol{u}}} \boldsymbol{u}_\epsilon(t).\end{aligned}$$

Since the nominal solution says $\tilde{\boldsymbol{y}}(t) = \boldsymbol{g}(t, \tilde{\boldsymbol{x}}(t), \tilde{\boldsymbol{u}}(t))$, then

$$\boldsymbol{y}_\epsilon(t) \approx \left.\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\tilde{\boldsymbol{x}},\, \boldsymbol{u}=\tilde{\boldsymbol{u}}} \boldsymbol{x}_\epsilon(t) + \left.\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}}\right|_{\boldsymbol{x}=\tilde{\boldsymbol{x}},\, \boldsymbol{u}=\tilde{\boldsymbol{u}}} \boldsymbol{u}_\epsilon(t).$$

Linearization (Step 2)

Step 2: Compute the four derivatives (called "Jacobians")

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix} \qquad \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}} = \begin{bmatrix} \frac{\partial g_1}{\partial x_1} & \cdots & \frac{\partial g_1}{\partial x_n} \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial x_1} & \cdots & \frac{\partial g_n}{\partial x_n} \end{bmatrix}$$

$$\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}} = \begin{bmatrix} \frac{\partial f_1}{\partial u_1} & \cdots & \frac{\partial f_1}{\partial u_p} \\ \vdots & & \vdots \\ \frac{\partial f_n}{\partial u_1} & \cdots & \frac{\partial f_n}{\partial u_p} \end{bmatrix} \qquad \frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}} = \begin{bmatrix} \frac{\partial g_1}{\partial u_1} & \cdots & \frac{\partial g_1}{\partial u_p} \\ \vdots & & \vdots \\ \frac{\partial g_n}{\partial u_1} & \cdots & \frac{\partial g_n}{\partial u_p} \end{bmatrix}$$

and evaluate each at the nominal solution $\boldsymbol{x}(t) = \tilde{\boldsymbol{x}}(t)$, $\boldsymbol{u}(t) = \tilde{\boldsymbol{u}}(t)$.

Linearization (Steps 3-4)

Step 3: Set

$$\begin{aligned}
\boldsymbol{A}(t) &= \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\tilde{\boldsymbol{x}},\, \boldsymbol{u}=\tilde{\boldsymbol{u}}} \\
\boldsymbol{B}(t) &= \left.\frac{\partial \boldsymbol{f}}{\partial \boldsymbol{u}}\right|_{\boldsymbol{x}=\tilde{\boldsymbol{x}},\, \boldsymbol{u}=\tilde{\boldsymbol{u}}} \\
\boldsymbol{C}(t) &= \left.\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{x}}\right|_{\boldsymbol{x}=\tilde{\boldsymbol{x}},\, \boldsymbol{u}=\tilde{\boldsymbol{u}}} \\
\boldsymbol{D}(t) &= \left.\frac{\partial \boldsymbol{g}}{\partial \boldsymbol{u}}\right|_{\boldsymbol{x}=\tilde{\boldsymbol{x}},\, \boldsymbol{u}=\tilde{\boldsymbol{u}}}
\end{aligned}$$

Step 4: Put it all together:

$$\begin{aligned}
\dot{\boldsymbol{x}}_\epsilon(t) &= \boldsymbol{A}(t)\boldsymbol{x}_\epsilon(t) + \boldsymbol{B}(t)\boldsymbol{u}_\epsilon(t) \\
\boldsymbol{y}_\epsilon(t) &= \boldsymbol{C}(t)\boldsymbol{x}_\epsilon(t) + \boldsymbol{D}(t)\boldsymbol{u}_\epsilon(t)
\end{aligned}$$

This is a SS description of a LTI/LTV, causal, lumped system. Solutions are only accurate in the "neighborhood" of $\tilde{\boldsymbol{x}}(t)$, $\tilde{\boldsymbol{u}}(t)$, and $\tilde{\boldsymbol{y}}(t)$.

A Remark on Linearization

The functions $\boldsymbol{f}$ and $\boldsymbol{g}$ *must be smooth* in the neighborhood of the solution $\tilde{\boldsymbol{x}}(t)$, $\tilde{\boldsymbol{u}}(t)$, and $\tilde{\boldsymbol{y}}(t)$, otherwise linearization is not going to be useful (the Taylor series approximations in the derivation won't hold).

Here is an example of a state-dynamic function that definitely isn't smooth at $x(t) = 0$:

$$\dot{x}(t) \quad = \quad \mathrm{sign}(x(t)) + u(t)$$

where

$$\mathrm{sign}(x) \quad = \quad \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$$

It should be clear that $\tilde{x}(t) = 0$ and $\tilde{u}(t) = 0$ is a solution. But the necessary derivatives don't exist here. Can't linearize.

Conclusions

- We're done discussing the different mathematical descriptions of systems. You should now understand:
  - Continuous-time and discrete-time I/O differential/difference equations.
  - Continuous-time and discrete-time transfer functions.
  - Continuous-time and discrete-time impulse responses.
  - Continuous-time and discrete-time state-space representations.
  - Capabilities and limitations of different mathematical descriptions.
  - How to move between different mathematical descriptions when a system is linear, time-invariant, causal, and lumped.
  - Time-varying state-space system description.
  - Linearization of nonlinear state-space system descriptions.
- Along the way, you had to learn a little bit of linear algebra:
  - Identity matrix
  - Matrix inverse
  - Determinant
  - Adjoint
- Next week: Begin analysis of linear state-space system descriptions.