ECE504: Lecture 4

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29-Sep-2009

Lecture 4 Major Topics

We are now starting Part II of ECE504: **Quantitative and qualitative analysis of systems**

mathematical description → results about behavior of system

Today:

1. Solution of LTI/LTV state equations for discrete-time systems
2. Solution of LTI/LTV state equations for continuous-time systems
3. Examples

You should be reading Chen Chapter 4 now. You should also refer back to Chen 3.2-3.3 to learn about "basis", "linear independence", and solutions to linear algebraic equations like $\boldsymbol{Ax} = \boldsymbol{y}$.

Linear State-Space Description of Discrete-Time Systems

$$\begin{aligned} \boldsymbol{x}[k+1] &= \boldsymbol{A}[k]\boldsymbol{x}[k] + \boldsymbol{B}[k]\boldsymbol{u}[k] \\ \boldsymbol{y}[k] &= \boldsymbol{C}[k]\boldsymbol{x}[k] + \boldsymbol{D}[k]\boldsymbol{u}[k] \end{aligned}$$

We assume a general model with p inputs, q outputs, and n states.

Given an initial time $k_0 \in \mathbb{Z}$, an initial state $\boldsymbol{x}[k_0] \in \mathbb{R}^n$, how does the state evolve for $k = k_0 + 1, k_0 + 2, \dots$?

Solution to State Equation

Following our induction, for all $k \geq k_0$, we can write

$$\boldsymbol{x}[k] = \boldsymbol{\Phi}[k,k_0]\boldsymbol{x}[k_0] + \sum_{\ell=k_0}^{k-1} \boldsymbol{\Phi}[k,\ell+1]\boldsymbol{B}[\ell]\boldsymbol{u}[\ell]$$

where $\boldsymbol{\Phi}$ is an $n \times n$ matrix valued function with two time arguments:

$$\boldsymbol{\Phi}[k,j] = \begin{cases} \text{undefined} & k < j \\ \boldsymbol{I}_n & k = j \\ \boldsymbol{A}[k-1]\boldsymbol{A}[k-2]\cdots\boldsymbol{A}[j] & k > j \end{cases}$$

Remarks:

- $\boldsymbol{I}_n$ is the $n \times n$ identity matrix.
- The order of the product $\boldsymbol{A}[k-1]\boldsymbol{A}[k-2]\cdots\boldsymbol{A}[j]$ is important because matrices don't usually commute.
- The matrix function $\boldsymbol{\Phi} : \mathbb{Z}^2 \mapsto \mathbb{R}^{n\times n}$ is called the state transition matrix (STM) corresponding to $\boldsymbol{A}[k]$.

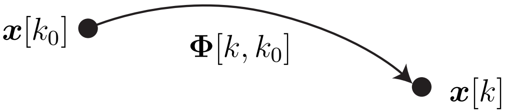
Zero-Input Response

Recall that linear systems have the nice property that we can separately analyze the zero-input response and the zero-state response.

Zero-input response: Given $\boldsymbol{u}[k] = 0$ for all $k \geq k_0$, we can write

$$\boldsymbol{x}[k] \;=\; \boldsymbol{\Phi}[k, k_0]\boldsymbol{x}[k_0]$$

The state transition matrix $\boldsymbol{\Phi}[k, k_0]$ describes how the state at time k_0 evolves to the state at time $k \geq k_0$ (in the absence of an input).



If the STM $\boldsymbol{\Phi}[k, k_0]$ is invertible, then $\boldsymbol{\Phi}^{-1}[k, k_0] = \boldsymbol{\Phi}[k_0, k]$. But there is no guarantee that it is invertible. This operation is one-way.

Zero-State Response

Zero-state response: Given $\boldsymbol{x}[k_0] = 0$, we can write

$$\boldsymbol{x}[k] = \sum_{\ell=k_0}^{k-1} \boldsymbol{\Phi}[k, \ell+1]\boldsymbol{B}[\ell]\boldsymbol{u}[\ell]$$

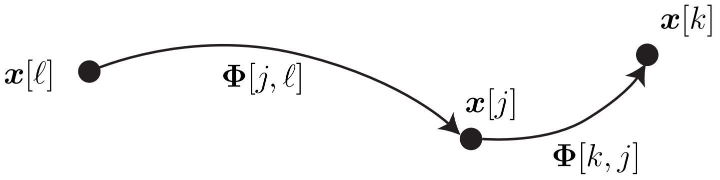
In this case, we have to compute several state transition matrices: $\boldsymbol{\Phi}[k, k_0+1], \boldsymbol{\Phi}[k, k_0+2], \ldots, \boldsymbol{\Phi}[k, k]$.

This looks like it might require a lot of computation as k gets larger. Fortunately, there are some nice properties of the state transition matrix that can ease the computational burden...

Some Basic Properties of the State Transition Matrix

1. $\boldsymbol{\Phi}[j,j] = \boldsymbol{I}_n$ for all $j \in \mathbb{Z}$.
2. $\boldsymbol{\Phi}[k+1,j] = \boldsymbol{A}[k]\boldsymbol{\Phi}[k,j]$ for all $k \geq j$.
3. If $\ell \leq j \leq k$, then $\boldsymbol{\Phi}[k,\ell] = \boldsymbol{\Phi}[k,j]\boldsymbol{\Phi}[j,\ell]$.

This last property is called the "semigroup" property. It intuitively says that the transition from $\boldsymbol{x}[\ell]$ to $\boldsymbol{x}[k]$ is the same as the transition from $\boldsymbol{x}[\ell]$ to $\boldsymbol{x}[j]$ followed by the transition from $\boldsymbol{x}[j]$ to $\boldsymbol{x}[k]$.



Special Case: $\boldsymbol{A}[k] \equiv \boldsymbol{A}$ for all $k \geq k_0$

When $\boldsymbol{A}[k] \equiv \boldsymbol{A}$ for all $k \geq k_0$, the product

$$\boldsymbol{A}[k-1]\boldsymbol{A}[k-2]\cdots\boldsymbol{A}[j] \;=\; \boldsymbol{A}\boldsymbol{A}\cdots\boldsymbol{A}$$

How many $\boldsymbol{A}$'s are involved in this product? ______

Hence, when $\boldsymbol{A}[k] \equiv \boldsymbol{A}$ for all $k \geq k_0$, the state transition matrix can be written as

$$\boldsymbol{\Phi}[k,j] \;=\; \begin{cases} \text{undefined} & k < j \\ \boldsymbol{I}_n & k = j \\ \boldsymbol{A}^{k-j} & k > j. \end{cases}$$

In this case, the solution to the DT state-update difference equation is

$$\boldsymbol{x}[k] = \boldsymbol{A}^{k-k_0}\boldsymbol{x}[k_0] + \sum_{\ell=k_0}^{k-1} \boldsymbol{A}^{k-\ell-1}\boldsymbol{B}[\ell]\boldsymbol{u}[\ell]$$

for all $k \geq k_0$.

Discrete-Time Output Solution

For all $k \ge k_0$, we can just plug our solution to the state equation into our state-space output equation to get

$$\boldsymbol{y}[k] = \underbrace{\boldsymbol{C}[k]\boldsymbol{\Phi}[k,k_0]\boldsymbol{x}[k_0]}_{\text{zero-input response}} + \underbrace{\boldsymbol{C}[k]\sum_{\ell=k_0}^{k-1}\boldsymbol{\Phi}[k,\ell+1]\boldsymbol{B}[\ell]\boldsymbol{u}[\ell] + \boldsymbol{D}[k]\boldsymbol{u}[k]}_{\text{zero-state response}}$$

If the system is LTI, then we can write

$$\boldsymbol{y}[k] = \underbrace{\boldsymbol{C}\boldsymbol{A}^{k-k_0}\boldsymbol{x}[k_0]}_{\text{zero-input response}} + \underbrace{\boldsymbol{C}\sum_{\ell=k_0}^{k-1}\boldsymbol{A}^{k-\ell-1}\boldsymbol{B}\boldsymbol{u}[\ell] + \boldsymbol{D}\boldsymbol{u}[k]}_{\text{zero-state response}}$$

Remarks on Discrete-Time State-Space Solutions

For causal, linear, lumped discrete-time systems with p input terminals, q output terminals, and n states, we have shown that, given $\boldsymbol{x}[k_0]$ and $\boldsymbol{u}[k]$ for all $k \geq k_0$, **there exists a unique solution** to the discrete-time state-update difference equation

$$\boldsymbol{x}[k+1] \quad = \quad \boldsymbol{A}[k]\boldsymbol{x}[k] + \boldsymbol{B}[k]\boldsymbol{u}[k].$$

That solution is

$$\boldsymbol{x}[k] \quad = \quad \boldsymbol{\Phi}[k, k_0]\boldsymbol{x}[k_0] + \sum_{\ell=k_0}^{k-1} \boldsymbol{\Phi}[k, \ell+1]\boldsymbol{B}[\ell]\boldsymbol{u}[\ell]$$

for all $k \geq k_0$ with $\boldsymbol{\Phi}[k, j]$ as defined earlier.

This also implies that, given $\boldsymbol{x}[k_0]$ and $\boldsymbol{u}[k]$ for all $k \geq k_0$, **there exists a unique solution** to the discrete-time output equation.

Discrete-Time State-Space Example

Continuous-Time Linear Systems

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) + \boldsymbol{B}(t)\boldsymbol{u}(t) \quad (1)$$

$$\boldsymbol{y}(t) = \boldsymbol{C}(t)\boldsymbol{x}(t) + \boldsymbol{D}(t)\boldsymbol{u}(t) \quad (2)$$

Theorem

For any $t_0 \in \mathbb{R}$, any $\boldsymbol{x}(t_0) \in \mathbb{R}^n$, and any $\boldsymbol{u}(t) \in \mathbb{R}^p$ for all $t \geq t_0$, there exists a unique solution $\boldsymbol{x}(t)$ for all $t \in \mathbb{R}$ to the state-update differential equation (1). It is given as

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{\Phi}(t, \tau)\boldsymbol{B}(\tau)\boldsymbol{u}(\tau)\, d\tau \quad t \in \mathbb{R}$$

where $\boldsymbol{\Phi}(t, s) : \mathbb{R}^2 \mapsto \mathbb{R}^{n \times n}$ is the unique function satisfying

$$\frac{d}{dt}\boldsymbol{\Phi}(t, s) = \boldsymbol{A}(t)\boldsymbol{\Phi}(t, s) \text{ with } \boldsymbol{\Phi}(s, s) = \boldsymbol{I}_n.$$

Theorem Remarks

- Note that this theorem claims two things:
  1. A solution to the state-update equation always **exists**.
  2. The solution is **unique**.
- Does every differential equation have a **unique solution**?
  - What about
  $$\dot{x}(t) = \frac{1}{t} \text{ with } x(0) = 5$$
  - What about
  $$\dot{x}(t) = 3(x(t))^{2/3} \text{ with } x(0) = 0$$
- Proof sketch:
  1. Establish existence constructively by giving a solution and showing that it satisfies the state-update equation.
  2. Establish uniqueness by showing that, given two solutions to the state-update equation, they must be identical.
- We will only do the first part of the proof. Please refer to Chen or any other good textbook for a proof of the second part.

Theorem: Existence Proof Warmup #1

An important skill in research is to develop intuition by looking at the simplest possible case. What is the simplest possible case for the continuous-time state dynamics equation? Let's first assume that everything is scalar, i.e. $p = q = n = 1$. Our state update equation becomes

$$\dot{x}(t) \quad = \quad a(t)x(t) + b(t)u(t)$$

Let

$$\phi(t,s) \quad := \quad \exp\left\{\int_s^t a(\tau)\,d\tau\right\}$$

What is $\phi(s,s)$?

What is $\frac{d}{dt}\phi(t,s)$?

Theorem: Existence Proof Warmup #1

Note that $\phi(t,s) = \exp\left\{\int_s^t a(\tau)\,d\tau\right\}$ always exists and satisfies its own differential equation:

$$\frac{d}{dt}\phi(t,s) = a(t)\phi(t,s) \text{ with } \phi(s,s) = 1.$$

Now let's try the following solution to the scalar state-update differential equation with initial state condition $x(t_0)$:

$$x(t) = \phi(t,t_0)x(t_0) + \int_{t_0}^{t} \phi(t,\tau)b(\tau)u(\tau)\,d\tau \qquad \forall t \in \mathbb{R}$$

To see that this is indeed a solution, we need to confirm two things:

1. Does our solution satisfy the initial condition requirement of the scalar state-update DE?
2. Does our solution really solve the scalar state-update DE?

Theorem: Existence Proof Warmup #2

To develop additional intuition, let's now assume that everything is time-invariant, i.e. $\boldsymbol{A}(t) \equiv \boldsymbol{A}$ and $\boldsymbol{B}(t) \equiv \boldsymbol{B}$. Our state update equation becomes

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}\boldsymbol{x}(t) + \boldsymbol{B}\boldsymbol{u}(t)$$

Let

$$\boldsymbol{\Phi}(t,s) := \sum_{k=0}^{\infty} \boldsymbol{A}^k \frac{1}{k!}(t-s)^k$$

What is $\boldsymbol{\Phi}(s,s)$?

What is $\frac{d}{dt}\boldsymbol{\Phi}(t,s)$?

Theorem: Existence Proof Warmup #2

Note that $\boldsymbol{\Phi}(t,s) = \sum_{k=0}^{\infty} \boldsymbol{A}^k \frac{1}{k!}(t-s)^k$ exists for any $\boldsymbol{A} \in \mathbb{R}^{n \times n}$, $t \in \mathbb{R}$, and $s \in \mathbb{R}$. Moreover, $\boldsymbol{\Phi}(t,s)$ satisfies its own differential equation:

$$\frac{d}{dt}\boldsymbol{\Phi}(t,s) = \boldsymbol{A}\boldsymbol{\Phi}(t,s) \text{ with } \boldsymbol{\Phi}(s,s) = \boldsymbol{I}_n.$$

Now let's try the following solution to the time-invariant matrix state-update differential equation with initial state condition $\boldsymbol{x}(t_0)$:

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t,t_0)\boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{\Phi}(t,\tau)\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau \qquad \forall t \in \mathbb{R}$$

To see that this is indeed a solution, we need to confirm two things:

1. Does our solution satisfy the initial condition requirement of the time-invariant matrix state-update DE?
2. Does our solution really solve the time-inv. matrix state-update DE?

Theorem: Existence Proof for General Case

For the general (non-scalar, time-varying) case, we propose the solution

$$\boldsymbol{x}(t) = \boldsymbol{\Phi}(t, t_0)\boldsymbol{x}(t_0) + \int_{t_0}^{t} \boldsymbol{\Phi}(t, \tau)\boldsymbol{B}(\tau)\boldsymbol{u}(\tau)\, d\tau \tag{3}$$

where the state transition matrix satisfies the matrix differential equation

$$\frac{d}{dt}\boldsymbol{\Phi}(t, s) = \boldsymbol{A}(t)\boldsymbol{\Phi}(t, s) \text{ with } \boldsymbol{\Phi}(s, s) = \boldsymbol{I}_n. \tag{4}$$

Note that (4) is consistent with our two warmup cases.

To complete the existence proof, we need to:

1. Show that (3) with $\boldsymbol{\Phi}(t, s)$ defined according to (4) satisfies the initial condition requirement of the state-update DE.
2. Show that (3) with $\boldsymbol{\Phi}(t, s)$ defined according to (4) is indeed a solution to the state-update DE.
3. Show that there always exists a solution to the matrix DE (4).

Theorem: Existence Proof for General Case: Part 1

Show that (3) with $\mathbf{\Phi}(t, s)$ defined according to (4) satisfies the initial condition requirement of the state-update DE.

Theorem: Existence Proof for General Case: Part 2

Show that (3) with $\mathbf{\Phi}(t, s)$ defined according to (4) is indeed a solution to the state-update DE.

Theorem: Existence Proof for General Case: Part 3

Show that there always exists a solution to the matrix DE (4).

Peano-Baker Series Example

$$\boldsymbol{A}(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix}$$

Fundamental Matrix Method

While the Peano-Baker series establishes existence (and thus concludes the proof of the existence and uniqueness theorem), it is sometimes easier to find $\mathbf{\Phi}(t,s)$ via the "fundamental matrix method" (Chen section 4.5).

Basic idea:

1. Consider the the continuous time DE with $\boldsymbol{x}(t) \in \mathbb{R}^n$
$$\dot{\boldsymbol{x}}(t) = \boldsymbol{A}(t)\boldsymbol{x}(t) \tag{5}$$
2. Choose n different initial conditions $\boldsymbol{x}_1(t_0), \dots, \boldsymbol{x}_n(t_0)$. These n initial condition vectors must be linearly independent.
3. These n different initial conditions lead to n different solutions to (5). Call these solutions $\boldsymbol{x}_1(t), \dots, \boldsymbol{x}_n(t)$ and put them into a matrix $\boldsymbol{X}(t) = [\boldsymbol{x}_1(t), \dots, \boldsymbol{x}_n(t)] \in \mathbb{R}^{n \times n}$.
4. Note that $\dot{\boldsymbol{X}}(t) = \boldsymbol{A}(t)\boldsymbol{X}(t)$. The quantity $\boldsymbol{X}(t)$ is called a fundamental matrix of (5). Is the fundamental matrix unique?

Fundamental Matrix Method

Let $\boldsymbol{X}(t)$ be any fundamental matrix of (5). Note that $\boldsymbol{X}(t)$ is invertible for all t (see Chen p. 107). The state transition matrix $\boldsymbol{\Phi}(t,s)$ can then be computed as

$$\boldsymbol{\Phi}(t,s) = \boldsymbol{X}(t)\boldsymbol{X}^{-1}(s).$$

Check:

$$\boldsymbol{\Phi}(s,s) =$$

$$\frac{d}{dt}\boldsymbol{\Phi}(t,s) =$$

Fundamental Matrix Example

$$\boldsymbol{A}(t) = \begin{bmatrix} 0 & 0 \\ t & 0 \end{bmatrix}$$

Remarks on the CT State-Transition Matrix $\mathbf{\Phi}(t,s)$

1. There are many ways to compute $\mathbf{\Phi}(t,s)$. Some are easier than others, but computing $\mathbf{\Phi}(t,s)$ is almost always difficult.
2. Do different methods for computing $\mathbf{\Phi}(t,s)$ lead to different solutions?
3. Unlike the DT-STM $\mathbf{\Phi}[k,j]$, the CT-STM $\mathbf{\Phi}(t,s)$ is defined for any $(t,s) \in \mathbb{R}^2$. This means that we can specify an initial state $\boldsymbol{x}(t_0)$ and compute the system response at times **prior** to t_0.
4. It is easy to show that $\mathbf{\Phi}(t,s)$ possesses the semi-group property, i.e.

$$\mathbf{\Phi}(t,\tau) = \mathbf{\Phi}(t,s)\mathbf{\Phi}(s,\tau)$$

for any $(t,\tau,s) \in \mathbb{R}^3$ from the fundamental matrix formulation:

$$\mathbf{\Phi}(t,\tau) = \mathbf{\Phi}(t,s)\mathbf{\Phi}(s,\tau) = \boldsymbol{X}(t)\boldsymbol{X}^{-1}(s)\boldsymbol{X}(s)\boldsymbol{X}^{-1}(\tau) = \boldsymbol{X}(t)\boldsymbol{X}^{-1}(\tau)$$

Important Special Case: $A(t) \equiv A$

When $\boldsymbol{A}(t) \equiv \boldsymbol{A}$, the state-transition matrix Peano-Baker series becomes

$$\begin{aligned}
\boldsymbol{\Phi}(t,s) &= \sum_{k=0}^{\infty} \boldsymbol{M}_k(t,s) \\
&= \sum_{k=0}^{\infty} \int_s^t \int_s^{\tau_1} \cdots \int_s^{\tau_{k-1}} \underbrace{\boldsymbol{A}\boldsymbol{A}\cdots\boldsymbol{A}}_{k-\text{fold product}} \; d\tau_k \cdots d\tau_1 \\
&= \sum_{k=0}^{\infty} \boldsymbol{A}^k \int_s^t \int_s^{\tau_1} \cdots \int_s^{\tau_{k-1}} d\tau_k \cdots d\tau_1
\end{aligned}$$

To compute $\boldsymbol{M}_k(t,s)$, let's look at $k = 0, 1, 2, \ldots$ to see the pattern...

Important Special Case: $\boldsymbol{A}(t) \equiv \boldsymbol{A}$

By induction, we can show that

$$\boldsymbol{M}_k(t,s) \quad = \quad \boldsymbol{A}^k \frac{1}{k!}(t-s)^k$$

hence

$$\boldsymbol{\Phi}(t,s) = \sum_{k=0}^{\infty} \boldsymbol{A}^k \frac{1}{k!}(t-s)^k$$

which is consistent with our earlier result (warmup #2).

Suppose, for $x \in \mathbb{C}$, we have

$$f(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}$$

What is $f(x)$?

Matrix Exponential

Definition (Matrix Exponential)

Given $\boldsymbol{W} \in \mathbb{C}^{n \times n}$, the matrix exponential is defined as

$$\exp(\boldsymbol{W}) = \sum_{k=0}^{\infty} \frac{\boldsymbol{W}^k}{k!}$$

Note that the matrix exponential is not performed element-by-element, i.e.

$$\exp\left(\begin{bmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{bmatrix}\right) \neq \begin{bmatrix} e^{w_{11}} & e^{w_{12}} \\ e^{w_{21}} & e^{w_{22}} \end{bmatrix}$$

Matlab has a special function (`expm`) that computes matrix exponentials. Calling `exp(W)` will not give the same results as `expm(W)`.

Important Special Case: $A(t) \equiv A$

Putting it all together, when $\boldsymbol{A}(t) \equiv \boldsymbol{A}$, we can say that

$$\boldsymbol{\Phi}(t,s) = \sum_{k=0}^{\infty} \boldsymbol{A}^k \frac{1}{k!}(t-s)^k = \exp\left\{(t-s)\boldsymbol{A}\right\}$$

Then the solution to the LTI continuous-time state-update DE is

$$\boxed{\boldsymbol{x}(t) = \exp\left\{(t-t_0)\boldsymbol{A}\right\}\boldsymbol{x}(t_0) + \int_{t_0}^{t} \exp\left\{(t-\tau)\boldsymbol{A}\right\}\boldsymbol{B}(\tau)\boldsymbol{u}(\tau)\,d\tau}$$

and the output equation is

$$\boxed{\boldsymbol{y}(t) = \boldsymbol{C}(t)\exp\left\{(t-t_0)\boldsymbol{A}\right\}\boldsymbol{x}(t_0) + \boldsymbol{C}(t)\int_{t_0}^{t} \exp\left\{(t-\tau)\boldsymbol{A}\right\}\boldsymbol{B}(\tau)\boldsymbol{u}(\tau)\,d\tau + \boldsymbol{D}(t)\boldsymbol{u}(t)}$$

Contrast/Comparison Between CT and DT Solutions

Similarities

- CT and DT solutions have same "look".
- CT and DT solutions have state transition matrices with same intuitive properties, e.g. semigroup.

Differences

- In DT systems, $\boldsymbol{x}[k]$ is only defined for $k \geq k_0$ because the DT-STM $\boldsymbol{\Phi}[k, k_0]$ is only defined for $k \geq k_0$.
- In CT systems, $\boldsymbol{x}(t)$ is only defined for all $t \in \mathbb{R}$ because the CT-STM $\boldsymbol{\Phi}(t, t_0)$ is defined for all $(t, t_0) \in \mathbb{R}^2$.
- We didn't prove this, but the CT-STM $\boldsymbol{\Phi}(t, t_0)$ is always invertible. This is not true of the DT-STM $\boldsymbol{\Phi}[k, k_0]$.

Conclusions: What We Now Know

- We know how to **solve** discrete-time LTV and LTI systems. "Solve" means "write an analytical expression for $\boldsymbol{x}[k]$ and $\boldsymbol{y}[k]$ given $\boldsymbol{A}[k]$, $\boldsymbol{B}[k]$, $\boldsymbol{C}[k]$, $\boldsymbol{D}[k]$, and $\boldsymbol{x}[k_0]$".
- We know that solutions must exist and must be unique.

- We know how to **solve** continuous-time LTV and LTI systems. "Solve" means "write an analytical expression for $\boldsymbol{x}(t)$ and $\boldsymbol{y}(t)$ given $\boldsymbol{A}(t)$, $\boldsymbol{B}(t)$, $\boldsymbol{C}(t)$, $\boldsymbol{D}(t)$, and $\boldsymbol{x}(t_0)$".
- We know that solutions must exist and must be unique.
- We also know two ways to compute the state transition matrix.

- We know some of the properties of state transition matrices.
- We know differences between the DT-STM and the CT-STM.

Next Time

1. Linear algebra tools to lay foundation for analysis of $\boldsymbol{A}^k$ and $\exp(\boldsymbol{A})$:
   - Subspaces
   - Nullspace and range
   - Rank
   - Matrix invertibility equivalences
2. Efficient ways to analyze and compute $\boldsymbol{A}^k$ and $\exp(\boldsymbol{A})$.
3. More DT-LTI and CT-LTI examples.