ECE504: Lecture 8

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Lecture 9 Major Topics

We are still in Part II of ECE504: **Quantitative and qualitative analysis of systems**

mathematical description $\rightarrow$ results about behavior of system

Today, we will cover:

- External stability of LTI systems
- A "new" way to compute $\exp\{t\boldsymbol{A}\}$ and $\boldsymbol{A}^k$ for LTI systems.
- Reachability of DT systems
- Controllability of DT systems
- Observability of DT systems
- The Cayley-Hamilton theorem.

You should be reading Chen Chapters 5 and 6 now. Sections 5.1-5.2, and 5.5 all discuss external stability. Sections 6.1-6.3 discuss controllability and observability.

Internal Versus External Stability

Recall that there are two types of stability that we can discuss when we use state-space descriptions of dynamic systems:

1. Internal stability (are the states blowing up?)
2. External stability (is the output blowing up?)

In our example CT-LTI system,

$$\begin{aligned}
\dot{\boldsymbol{x}}(t) &= \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \boldsymbol{u}(t) \\
\boldsymbol{y}(t) &= \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \end{bmatrix} \boldsymbol{u}(t)
\end{aligned}$$

we saw a system that was externally stable, but not internally stable.

Bounded Input – Bounded Output (BIBO) Stability

Definition

A continuous-time system is BIBO stable if, for every input satisfying

$$|u(t)| \leq M_u$$

for all $t \in \mathbb{R}$ and some $0 \leq M_u < \infty$, the output satisfies

$$|y(t)| \leq M_y$$

for all $t \in \mathbb{R}$ and some $0 \leq M_y < \infty$.

Any bounded input causes the system to produce a bounded output.

Note that this definition is for SISO systems, but can easily be extended to MIMO systems using our notion of bounded vectors from Lecture 8.

CT-LTI Systems: First Criterion for BIBO Stability

Theorem (Chen Theorem 5.1)

A CT-LTI system with impulse response $g(t)$ is BIBO stable if and only if

$$\int_{-\infty}^{\infty} |g(t)|\, dt < \infty.$$

In other words, the impulse response of the system must be "absolutely integrable" for the system to be BIBO stable. The converse is also true.

Intuitive examples:

$$g(t) = \begin{cases} 0 & t < 0 \\ e^{-t} & t \geq 0 \end{cases} \qquad g(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0 \end{cases} \qquad g(t) = \begin{cases} 0 & t < 0 \\ \sin(t) & t \geq 0 \end{cases}$$

Proof of the First Criterion for BIBO Stability

CT-LTI Systems: Second Criterion for BIBO Stability

Theorem (Chen Theorem 5.3)

A CT-LTI system is BIBO stable if and only if all of the poles of its proper rational transfer function $\hat{g}(s)$ *have negative real parts.*

Recall that a proper rational transfer function is one in which the degree of the denominator is equal to or larger than the degree of the numerator.

Intuitive example: Suppose we had a system with transfer function:

$$\hat{g}(s) = \frac{1}{s^2+1}$$

Is this transfer function rational and proper? Is this system BIBO stable?

Why can't this system be BIBO stable? Hint: Is there a bounded input that causes the output of this system to "blow up"?

CT-LTI Systems: Third Criterion for BIBO Stability

Theorem

Given a CT-LTI system described by the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$, then the following facts are true:

1. *If $\boldsymbol{A}$ is Hurwitz, then the system is BIBO stable.*
2. *If the system is BIBO stable and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ is a* **minimal realization**, *then $\boldsymbol{A}$ is Hurwitz.*

Minimal realization (intuitive definition): The transfer function $\hat{g}(s)$ corresponding to the state space system described by the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ has no pole/zero cancellations.

1. Is every pole of the transfer function $\hat{g}(s)$ an eigenvalue of $\boldsymbol{A}$?
2. Is every eigenvalue of $\boldsymbol{A}$ a pole of the transfer function $\hat{g}(s)$?
3. If $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ is a minimal realization, is every eigenvalue of $\boldsymbol{A}$ a pole of the transfer function $\hat{g}(s)$?

CT-LTI Systems: Third Criterion for BIBO Stability

Proof sketch:

The proof of Fact 1 follows directly from the fact that every pole of the transfer function $\hat{g}(s)$ an eigenvalue of $\boldsymbol{A}$ and the Second Criterion for BIBO Stability.

The proof of Fact 2 follows directly from the fact that, if $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ is a minimal realization, then every eigenvalue of $\boldsymbol{A}$ a pole of the transfer function $\hat{g}(s)$ (and the Second Criterion for BIBO Stability).

Impulse Response from the CT-LTI SS Description

Recall that

$$y(t) \;=\; \boldsymbol{C}\boldsymbol{x}(t) + \boldsymbol{D}u(t)$$

When $\boldsymbol{x}(0) = \mathbf{0}$, the solution to the CT-LTI differential state equation is (see Chen Chap 4 or your lecture notes):

$$\boldsymbol{x}(t) \;=\; \int_0^t \exp\{(t-\tau)\boldsymbol{A}\}\boldsymbol{B}u(\tau)\,d\tau.$$

Hence, the zero-state response of a CT-LTI system can be computed from the state-space description as

$$y(t) \;=\; \int_0^t \boldsymbol{C}\exp\{(t-\tau)\boldsymbol{A}\}\boldsymbol{B}u(\tau)\,d\tau + \boldsymbol{D}u(t)$$

What is the impulse response of the system?

A "New" Way to Compute $\exp\{t\boldsymbol{A}\}$

We know that, for a CT-LTI system, the transfer function

$$\hat{g}(s) = \boldsymbol{C}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$$

We also know that

$$\begin{aligned}\hat{g}(s) &= \mathcal{L}\left[g(t)\right] \\ &= \mathcal{L}\left[\boldsymbol{C}\exp\{t\boldsymbol{A}\}\boldsymbol{B}\mathbb{I}(t) + \boldsymbol{D}\delta(t)\right]\end{aligned}$$

which implies that

$$\boxed{\exp\{t\boldsymbol{A}\} = \mathcal{L}^{-1}\left[(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\right]}$$

Example....

Bounded Input – Bounded Output (BIBO) Stability

Definition

A discrete-time system is BIBO stable if, for every input satisfying

$$|u[k]| \le M_u$$

for all $k \in \mathbb{Z}$ and some $0 \le M_u < \infty$, the output satisfies

$$|y[k]| \le M_y$$

for all $k \in \mathbb{Z}$ and some $0 \le M_y < \infty$.

Essentially the same definition as for continuous-time systems.

DT-LTI Systems: First/Second Criteria for BIBO Stability

Theorem

A DT-LTI system with impulse response $g[k]$ is BIBO stable if and only if

$$\sum_{k=-\infty}^{\infty} |g[k]|\, dt < \infty.$$

In other words, the impulse response of the system must be "absolutely summable" for the system to be BIBO stable. The converse is also true.

Theorem

A DT-LTI system is BIBO stable if and only if all of the poles of its proper rational transfer function $\hat{g}(z)$ have magnitude less than one.

DT-LTI Systems: Third Criterion for BIBO Stability

Theorem

Given a DT-LTI system described by the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$, then the following facts are true:

1. *If $\boldsymbol{A}$ is Schur, then the system is BIBO stable.*
2. *If the system is BIBO stable and $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$ is a* **minimal realization**, *then $\boldsymbol{A}$ is Schur.*

Intuition and proof are essentially the same as the CT-LTI case.

Impulse Response from the DT-LTI SS Description

Recall that

$$y[k] \;\; = \;\; \boldsymbol{C}\boldsymbol{x}[k] + \boldsymbol{D}u[k]$$

When $\boldsymbol{x}[0] = \boldsymbol{0}$, the solution to the DT-LTI differential state equation is

$$\boldsymbol{x}[k] \;\; = \;\; \sum_{\ell=0}^{k-1} \boldsymbol{A}^{k-\ell-1}\boldsymbol{B}u[\ell]$$

Hence, the zero-state response of a DT-LTI system can be computed from the state-space description as

$$y[k] \;\; = \;\; \sum_{\ell=0}^{k-1} \boldsymbol{C}\boldsymbol{A}^{k-\ell-1}\boldsymbol{B}u[\ell] + \boldsymbol{D}u[k]$$

What is the impulse response of the system?

A "New" Way to Compute A^k

We know that, for a DT-LTI system, the transfer function

$$\hat{g}(z) = \boldsymbol{C}(z\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}$$

We also know that

$$\begin{aligned}\hat{g}(z) &= \mathcal{Z}\left[g[k]\right] \\ &= \mathcal{Z}\left[\boldsymbol{C}\boldsymbol{A}^{k-1}\boldsymbol{B}\mathbb{I}[k] + \boldsymbol{D}\delta[k]\right]\end{aligned}$$

which implies that

$$\boxed{\boldsymbol{A}^{k-1} = \mathcal{Z}^{-1}\left[(z\boldsymbol{I}_n - \boldsymbol{A})^{-1}\right]}$$

Example....

Reachability, Controllability, and Observability

Recall our intuitive concept of the CT "state" $\boldsymbol{x}(t_0)$: everything you need to know about the system at time $t = t_0$ to compute the outputs $y(t)$ for all $t \geq t_0$ given the inputs $u(t)$ for all $t \geq t_0$.

The same concept applies to DT systems.

In many real systems, the number of states tends to be much larger than the number of inputs and the number of outputs.

Intuitively, this suggests that

- You can't do much to the state $\boldsymbol{x}$ by manipulating the input $\boldsymbol{u}$
- You can't determine much about the state $\boldsymbol{x}$ by observing the output $\boldsymbol{y}$

But is this intuition true?

Reachability, Controllability, and Observability

A quick example:

$$\dot{\boldsymbol{x}}(t) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \boldsymbol{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

Review question: Given the initial state $\boldsymbol{x}(0)$ and the input $u(t)$ for all $t \geq t_0$, what is $\boldsymbol{x}(t)$?

What influence does $u(t)$ have on the state $x_1(t)$?

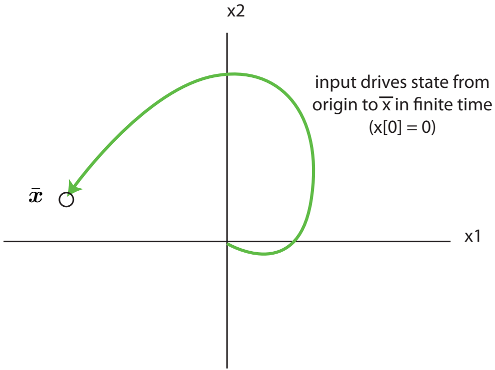
What influence does $u(t)$ have on the state $x_2(t)$?

Now suppose $\boldsymbol{C} = [0\ 1]$, i.e. $y(t) = x_2(t)$. What does the output tell us about the state $x_1(t)$? Is this true for any $\boldsymbol{A}$?

Reachable States (DT Systems)

Definition

The state $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ is a **reachable state** if there exists $k_r > 0$ and an input sequence $\{u[0], u[1], \dots, u[k_r - 1]\}$ such that $\boldsymbol{x}(k_r) = \bar{\boldsymbol{x}}$ when $\boldsymbol{x}[0] = \mathbf{0}$ and when you apply the chosen input sequence $\{u[0], u[1], \dots, u[k_r - 1]\}$.



Reachability Matrix for DT-LTI Systems

Definition

Given a DT-LTI system described by the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$, the **reachability matrix** of this system is the matrix

$$\boldsymbol{Q}_r = [\boldsymbol{B}\ \boldsymbol{A}\boldsymbol{B}\ \cdots\ \boldsymbol{A}^{n-1}\boldsymbol{B}]$$

What are the dimensions of $\boldsymbol{Q}_r$ if we have a DT-LTI system with p inputs?

Reachability Theorem for DT-LTI Systems

Theorem

$\bar{\boldsymbol{x}}$ is a reachable state if and only if $\bar{\boldsymbol{x}} \in \mathrm{range}(\boldsymbol{Q}_r)$.

Example:

$$\boldsymbol{A} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Is $\bar{\boldsymbol{x}} = [1, 0]^\top$ reachable? How about $\bar{\boldsymbol{x}} = [9, 9]^\top$?

To prove the reachability theorem, we are going to need to take a brief detour and learn an important linear algebra result...

Detour: The Cayley-Hamilton Theorem

Theorem

Given $\boldsymbol{A} \in \mathbb{R}^{n\times n}$*, suppose that*

$$\begin{aligned}\det(\lambda \boldsymbol{I}_n - \boldsymbol{A}) &= (\lambda-\lambda_1)^{r_1}(\lambda-\lambda_2)^{r_2}\cdots(\lambda-\lambda_s)^{r_s} \\ &= \lambda^n + a_1\lambda^{n-1} + \cdots + a_n.\end{aligned}$$

Then the matrix $\boldsymbol{A}$ **satisfies its own characteristic polynomial** *in the sense that*

$$\boldsymbol{A}^n + a_1\boldsymbol{A}^{n-1} + \cdots + a_n\boldsymbol{I}_n = \boldsymbol{0}_{n\times n}$$

or, equivalently,

$$(\boldsymbol{A}-\lambda_1\boldsymbol{I}_n)^{r_1}(\boldsymbol{A}-\lambda_2\boldsymbol{I}_n)^{r_2}\cdots(\boldsymbol{A}-\lambda_s\boldsymbol{I}_n)^{r_s} = \boldsymbol{0}_{n\times n}.$$

An Example of the Cayley-Hamilton Theorem

$$\boldsymbol{A} = \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$$

Cayley-Hamilton Theorem Proof Sketch

Proof of the Reachability Theorem for DT-LTI Systems

Remarks on the Reachability Theorem

Interesting consequence: The reachability definition requires that we drive the state from the origin to $\bar{\boldsymbol{x}}$ in a finite number of steps. The reachability theorem implies that we can always drive the state from $\mathbf{0}$ to $\bar{\boldsymbol{x}} \in \mathrm{range}(\boldsymbol{Q}_r)$ in n steps (or less).

Fact: The set of reachable states is equal to $\mathrm{range}(\boldsymbol{Q}_r)$ and is a subspace of $\mathbb{R}^n$.

Definition

An LTI system with $\mathrm{range}(\boldsymbol{Q}_r) = \mathbb{R}^n$, i.e. all states are reachable, is called a "reachable" system.

Extensions of the Reachability Theorem

Theorem

The set of reachable states $\mathrm{range}(\boldsymbol{Q}_r)$ *is invariant under* $\boldsymbol{A}$*, i.e. if* $\boldsymbol{x} \in \mathrm{range}(\boldsymbol{Q}_r)$ *then* $\boldsymbol{A}\boldsymbol{x} \in \mathrm{range}(\boldsymbol{Q}_r)$*.*

The proof is a consequence of the Cayley-Hamilton theorem.

Theorem

If $\mathrm{range}(\boldsymbol{Q}_r) = \mathbb{R}^n$ *(all states are reachable states) for a SS description with matrices* $\boldsymbol{A}$*,* $\boldsymbol{B}$*,* $\boldsymbol{C}$*, and* $\boldsymbol{D}$*, then* $\mathrm{range}(\boldsymbol{Q}_r) = \mathbb{R}^n$ *for a SS description with matrices* $\boldsymbol{P}\boldsymbol{A}\boldsymbol{P}^{-1}$*,* $\boldsymbol{P}\boldsymbol{B}$*,* $\boldsymbol{C}\boldsymbol{P}^{-1}$*, and* $\boldsymbol{D}$ *for any invertible* $\boldsymbol{P} \in \mathbb{R}^{n \times n}$*.*

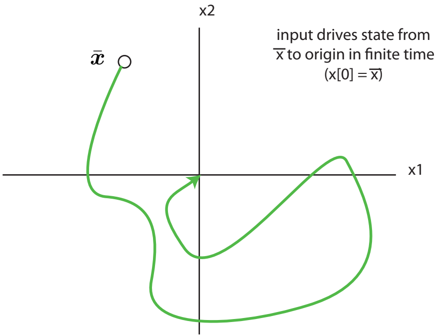
The proof can be found in Chen Theorem 6.2 (Chen discusses "controllability", rather than reachability, but the proof steps are identical).

DT-LTI Reachability Example

Controllability (DT systems)

Definition

The state $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ is a **controllable state** if there exists $k_r > 0$ and an input sequence $\{u[0], u[1], \dots, u[k_r - 1]\}$ such that $\boldsymbol{x}(k_r) = \mathbf{0}$ when $\boldsymbol{x}[0] = \bar{\boldsymbol{x}}$ and when you apply the chosen input sequence $\{u[0], u[1], \dots, u[k_r - 1]\}$.



Controllability vs. Reachability for DT-LTI Systems

Theorem

The set of reachable states is a subset of the set of controllable states for DT-LTI systems.

To see that this must be true, we will show that if $\bar{\boldsymbol{x}}$ is a reachable state, then it must also be a controllable state...

DT-LTI example of a case when the set of reachable states is not equal to the set of controllable states:

$$\boldsymbol{A} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \text{ and } \boldsymbol{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

Remarks on Controllability for DT-LTI Systems

Fact: The set of controllable states is a subspace of $\mathbb{R}^n$.

Theorem

The set of controllable states is invariant under $\boldsymbol{A}$, i.e. if $\boldsymbol{x}$ is in the set of controllable states, then so is $\boldsymbol{A}\boldsymbol{x}$.

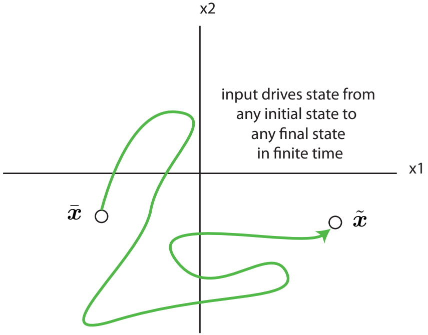
Definition

An LTI system with the set of controllable states equal to $\mathbb{R}^n$, i.e. all states are controllable, is called a "controllable" system.

It turns out that, like reachable states, if $\bar{\boldsymbol{x}}$ is a controllable state, then you can always find an input sequence $\{u[0], \dots, u[n-1]\}$ that drives the state from $\boldsymbol{x}[0] = \bar{\boldsymbol{x}}$ to the origin <u>in n or fewer steps</u>.

Chen's Definition of Controllable DT-LTI Systems

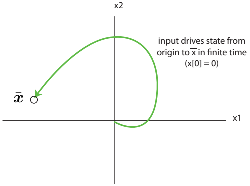
Note: Chen's definition of a controllable system is a system that can be driven from any initial state $\bar{\boldsymbol{x}}$ to any final state $\tilde{\boldsymbol{x}}$ in finite time.



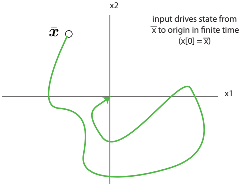
Is Chen's definition of a controllable system equivalent to our definition of a reachable system, our definition of a controllable system, or neither?

Summary: Reachability/Controllability for DT-LTI Systems

- Reachability: Drive state from $\mathbf{0}$ to $\bar{\boldsymbol{x}}$.



- Controllability: Drive state from $\bar{\boldsymbol{x}}$ to $\mathbf{0}$.



- $\{\text{reachable states}\} = \text{range}([\boldsymbol{B}\ \boldsymbol{AB}\ \cdots \boldsymbol{A}^{n-1}\boldsymbol{B}]) = \text{range}(\boldsymbol{Q}_r)$.
- $\{\text{reachable states}\} \subseteq \{\text{controllable states}\}$.
- Systems with more states typically take more time to control.

Observability for DT-LTI Systems

Definition

The state $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ is an **unobservable state** if, for any choice of input sequence $\{u[0], u[1], \dots\}$, the output sequence $\{y[0], y[1], \dots\}$ given initial $\boldsymbol{x}[0] = \bar{\boldsymbol{x}}$ is the same as the output sequence given initial $\boldsymbol{x}[0] = \mathbf{0}$.

Intuition:

- You want to determine the initial state of the system $\boldsymbol{x}[0]$.
- You are allowed to apply any input you want to the system and measure the resulting output.
- The state $\boldsymbol{x}[0] = \bar{\boldsymbol{x}}$ is called "unobservable" if you are unable to distinguish it from the initital state $\boldsymbol{x}[0] = \mathbf{0}$.

Observability for DT-LTI Systems

Given $\boldsymbol{x}[0] = \bar{\boldsymbol{x}}$, the output of a DT-LTI system is

$$y[k] = \boldsymbol{C}\boldsymbol{A}^k\bar{\boldsymbol{x}} + \sum_{\ell=0}^{k-1} \boldsymbol{C}\boldsymbol{A}^{k-\ell-1}\boldsymbol{B}\boldsymbol{u}[\ell] + \boldsymbol{D}\boldsymbol{u}[k]$$

Given $\boldsymbol{x}[0] = \mathbf{0}$, the output of a DT-LTI system is

$$y[k] = \mathbf{0} + \sum_{\ell=0}^{k-1} \boldsymbol{C}\boldsymbol{A}^{k-\ell-1}\boldsymbol{B}\boldsymbol{u}[\ell] + \boldsymbol{D}\boldsymbol{u}[k]$$

Hence, for these to be equal, we must have

$$\boxed{\boldsymbol{C}\boldsymbol{A}^k\bar{\boldsymbol{x}} = \mathbf{0} \qquad \forall k \geq 0}$$

This condition is equivalent to the statement that "$\bar{\boldsymbol{x}}$ is unobservable".

Observability Matrix for DT-LTI Systems

Definition

Given a DT-LTI SS system described by the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$, the **observability matrix** of the system is defined as

$$\boldsymbol{Q}_o = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \\ \vdots \\ \boldsymbol{C}\boldsymbol{A}^{n-1} \end{bmatrix}$$

What are the dimensions of $\boldsymbol{Q}_o$ if we have a system with q outputs?

Observability Theorem for DT-LTI Systems

Theorem

$\bar{\boldsymbol{x}}$ is an unobservable state if and only if $\bar{\boldsymbol{x}} \in \mathrm{null}(\boldsymbol{Q}_o)$.

Proof...

Definition

A DT-LTI system is **observable** if no $\bar{\boldsymbol{x}} \neq \boldsymbol{0}$ is an unobservable state.

In other words, a DT-LTI system is observable if $\dim(\mathrm{null}(\boldsymbol{Q}_o)) = 0$ or, equivalently, if $\mathrm{rank}(\boldsymbol{Q}_o) = n$.

Some possibly useful facts:

- The set of unobservable states is a subspace of $\mathbb{R}^n$.
- This subspace is invariant under $\boldsymbol{A}$, i.e. if $\bar{\boldsymbol{x}}$ is in the set of unobservable states, then so is $\boldsymbol{A}\boldsymbol{x}$.

Conclusions

Today we covered:

- BIBO stability for CT and DT systems with theorems for LTI systems.
- Reachability for DT systems.
- Controllability for DT systems.
- Observability for DT systems.
- Lots of theorems and definitions for DT-LTI systems.

Next time:

- Reachability, controllability, and observability for CT systems.
- Theorems and definitions for CT-LTI systems.
- Realization theory (Chen Chap 7).