ECE504: Lecture 9

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10-Nov-2009

Lecture 10 Major Topics

We are finishing up Part II of ECE504: **Quantitative and qualitative analysis of systems**

mathematical description → results about behavior of system

Today, we will cover:

- Reachability of CT systems
- Controllability of CT systems
- Observability of CT systems
- Introduction to realization theory
- Minimal realizations

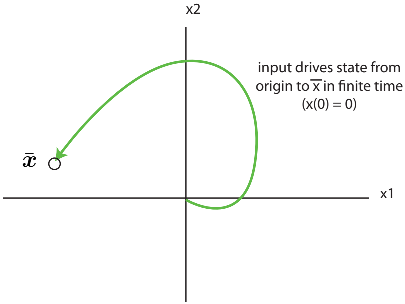
You should be reading Chen Chapters 6 and 7 now. Sections 6.1-6.3 discuss controllability and observability for CT systems. Sections 6.6-6.7 discuss controllability and observability for DT systems. Sections 7.1-7.2 give an introduction to realization theory.

Reachable States (CT Systems)

Definition

The state $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ is a **reachable state** if there exists $0 < T < \infty$ and an input function $\boldsymbol{u}(t)$ defined on $t \in [0, T]$ such that $\boldsymbol{x}(T) = \bar{\boldsymbol{x}}$ when $\boldsymbol{x}(0) = \mathbf{0}$ and when you apply the chosen input function $\boldsymbol{u}(t)$.

The idea of reachability in CT systems is the same as in DT systems. We want to drive the state from $\mathbf{0}$ to $\bar{\boldsymbol{x}}$ in finite time.



Reachability Matrix/Theorem for CT-LTI Systems

Definition

Given a CT-LTI system described by the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$, the **reachability matrix** of this system is the matrix

$$\boldsymbol{Q}_r = [\boldsymbol{B}\ \boldsymbol{A}\boldsymbol{B}\ \cdots\ \boldsymbol{A}^{n-1}\boldsymbol{B}]$$

The reachability matrix for CT-LTI systems is defined in exactly the same way as for DT-LTI systems. The reachability theorem is also identical:

Theorem

$\bar{\boldsymbol{x}}$ *is a reachable state if and only if* $\bar{\boldsymbol{x}} \in \operatorname{range}(\boldsymbol{Q}_r)$.

The proof for this result is not as easy as the DT-LTI case, but is illustrative nonetheless...

Proof Strategy — Preliminaries

First note that

$$[\exp\{t\boldsymbol{A}\}]^\top = \left[\sum_{k=0}^{\infty} \frac{t^k}{k!}\boldsymbol{A}^k\right]^\top = \sum_{k=0}^{\infty} \frac{t^k}{k!}\left[\boldsymbol{A}^\top\right]^k = \exp\{t\boldsymbol{A}^\top\}$$

Now define the "T-reachability Grammian" corresponding to $\boldsymbol{A}$ and $\boldsymbol{B}$

$$\boldsymbol{M}_r(T) := \int_0^T \exp\{(T-\tau)\boldsymbol{A}\}\boldsymbol{B}\boldsymbol{B}^\top \exp\{(T-\tau)\boldsymbol{A}^\top\}\, d\tau$$

What are the dimensions of $\boldsymbol{M}_r(T)$? _______

Does $\boldsymbol{M}_r(T)$ have any special properties?

Proof Strategy

We've defined

$$M_r(T) \;:=\; \int_0^T \exp\{(T-\tau)A\}BB^\top \exp\{(T-\tau)A^\top\}\,d\tau$$

$$Q_r = [B \; AB \; \cdots \; A^{n-1}B]$$

The strategy: For arbitrary $0 < T < \infty$, we will show

1. $\mathrm{range}(M_r(T)) \overset{(a)}{\subset}$ {reachable states} $\overset{(b)}{\subset} \mathrm{range}(Q_r)$
2. $\mathrm{range}(M_r(T)) \overset{(c)}{=} \mathrm{range}(Q_r)$

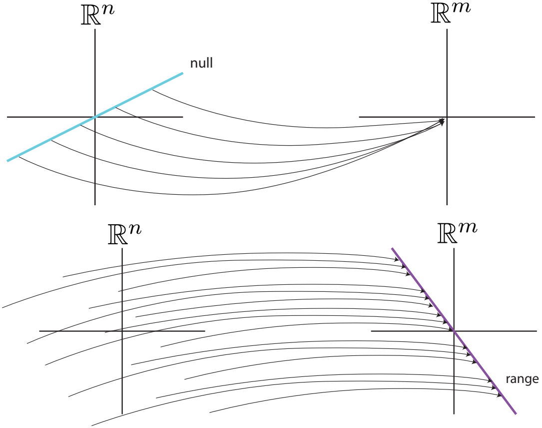
The second part implies the equality of two subspaces. By "sandwiching", the third subspace (the set of reachable states) must also be equal to the other two subspaces.

Bonus: We can use the Grammian to test for reachability.

Strategy Part 1 Proof (show (a) and (b))

Strategy Part 2 Proof — Preliminaries

This part is going to require two new linear algebra results. First, recall that, for $\boldsymbol{A} \in \mathbb{R}^{n \times m}$, $\mathrm{range}(\boldsymbol{A})$ is a subspace of $\boldsymbol{R}^n$ and $\mathrm{null}(\boldsymbol{A})$ is a subspace of $\boldsymbol{R}^m$. What can you say about $\mathrm{range}(\boldsymbol{A}^\top)$ and $\mathrm{null}(\boldsymbol{A}^\top)$?



Strategy Part 2 Proof — Preliminaries

Here are our two new linear algebra results:

Lemma

$\dim(\mathrm{range}(\boldsymbol{A})) + \dim(\mathrm{null}(\boldsymbol{A}^\top)) = n$ *for any* $\boldsymbol{A} \in \mathbb{R}^{n\times m}$.

Lemma

If $\boldsymbol{v} \in \mathrm{range}(\boldsymbol{A})$ *and* $\boldsymbol{w} \in \mathrm{null}(\boldsymbol{A}^\top)$, *then* $\boldsymbol{v}^\top \boldsymbol{w} = 0$.

What does this imply about $\mathrm{range}(\boldsymbol{A})$ and $\mathrm{null}(\boldsymbol{A}^\top)$?

Strategy Part 2 Proof (show (c))

Remarks on the Reachability Theorem

We now know that the set of reachable states is equal to $\mathrm{range}(\boldsymbol{Q}_r)$ and is a subspace of $\mathbb{R}^n$.

Interesting consequence: The reachability definition requires that we drive the state from the origin to $\bar{\boldsymbol{x}}$ in a finite time. The reachability theorem implies that we can always drive the state from $\mathbf{0}$ to $\bar{\boldsymbol{x}} \in \mathrm{range}(\boldsymbol{Q}_r)$ as quickly as we want in CT-LTI systems.

Theorem

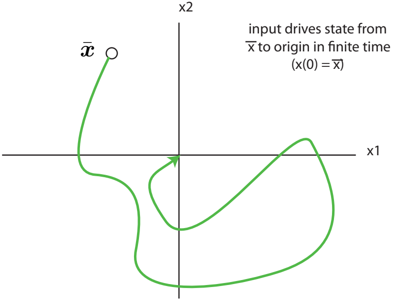
The set of reachable states $\mathrm{range}(\boldsymbol{Q}_r)$ *is invariant under* $\boldsymbol{A}$*, i.e. if* $\boldsymbol{x} \in \mathrm{range}(\boldsymbol{Q}_r)$ *then* $\boldsymbol{A}\boldsymbol{x} \in \mathrm{range}(\boldsymbol{Q}_r)$*.*

The proof is a consequence of the Cayley-Hamilton theorem.

Controllability versus Reachability for CT Systems

Definition

The state $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ is a **controllable state** if there exists $0 < T < \infty$ and an input function $u(t)$ defined on $t \in [0, T]$ such that $\boldsymbol{x}(T) = \mathbf{0}$ when $\boldsymbol{x}(0) = \bar{\boldsymbol{x}}$ and when you apply the chosen input function $u(t)$.



What can we say about the relationship between reachable states and controllable states for CT systems?

More Remarks on the Reachability Theorem

Lemma

$\bar{\boldsymbol{x}}$ is a controllable state if and only if $\bar{\boldsymbol{x}}$ is a reachable state.

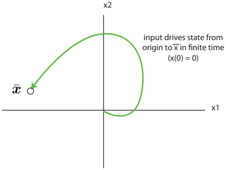
The set of reachable states is identical to the set of controllable states in CT-LTI systems because the CT-STM is always invertible. This is not true, as we saw, for DT-LTI systems.

Definition

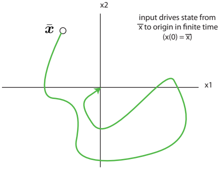
A CT-LTI system with $\mathrm{range}(\boldsymbol{Q}_r) = \mathbb{R}^n$, i.e. all states are reachable (or controllable), is called a "reachable" (or "controllable") system.

Summary: Reachability/Controllability for CT-LTI Systems

- Reachability: Drive state from $\mathbf{0}$ to $\bar{\boldsymbol{x}}$.



- Controllability: Drive state from $\bar{\boldsymbol{x}}$ to $\mathbf{0}$.



- $\{\text{reachable states}\} = \text{range}([\boldsymbol{B}\ \boldsymbol{A}\boldsymbol{B}\ \cdots\ \boldsymbol{A}^{n-1}\boldsymbol{B}]) = \text{range}(\boldsymbol{Q}_r)$.
- $\{\text{reachable states}\} = \{\text{controllable states}\}$.
- If $\bar{\boldsymbol{x}}$ and $\tilde{\boldsymbol{x}}$ are both reachable/controllable then we can find an input to drive the system from $\bar{\boldsymbol{x}}$ to $\tilde{\boldsymbol{x}}$ (or vice-versa) as quickly as we want.

Observability for CT-LTI Systems

Definition

The state $\bar{\boldsymbol{x}} \in \mathbb{R}^n$ is an **unobservable state** if, for any choice of input function $\boldsymbol{u}(t) : [0,\infty) \mapsto \mathbb{R}^p$, the output function $\boldsymbol{y}(t) : [0,\infty) \mapsto \mathbb{R}^q$ given initial state $\boldsymbol{x}(0) = \bar{\boldsymbol{x}}$ is the same as the output function given initial state $\boldsymbol{x}(0) = \boldsymbol{0}$.

Intuition:

- Basically the same idea as DT-LTI systems.
- You want to determine the intitial state of the system $\boldsymbol{x}(0)$.
- You are allowed to apply any input you want to the system and measure the resulting output.
- The state $\boldsymbol{x}(0) = \bar{\boldsymbol{x}}$ is called "unobservable" if you are unable to distinguish it from the initital state $\boldsymbol{x}(0) = \boldsymbol{0}$ for any input.

Observability for CT-LTI Systems

Given $\boldsymbol{x}(0) = \bar{\boldsymbol{x}}$, the output of a CT-LTI system is

$$y(t) = \boldsymbol{C}\exp\{t\boldsymbol{A}\}\bar{\boldsymbol{x}} + \int_0^t \boldsymbol{C}\exp\{(t-\tau)\boldsymbol{A}\}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau + \boldsymbol{D}\boldsymbol{u}(t)$$

Given $\boldsymbol{x}(0) = \boldsymbol{0}$, the output of a CT-LTI system is

$$y(t) = \int_0^t \boldsymbol{C}\exp\{(t-\tau)\boldsymbol{A}\}\boldsymbol{B}\boldsymbol{u}(\tau)\,d\tau + \boldsymbol{D}\boldsymbol{u}(t)$$

Hence, for these to be equal, we must have

$$\boxed{\boldsymbol{C}\exp\{t\boldsymbol{A}\}\bar{\boldsymbol{x}} = \boldsymbol{0} \qquad \forall t \geq 0}$$

This condition is equivalent to the statement that "$\bar{\boldsymbol{x}}$ is unobservable".

Observability Matrix/Theorem for CT-LTI Systems

Definition

Given a CT-LTI SS system described by the matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$, and $\boldsymbol{D}$, the **observability matrix** of the system is defined as

$$\boldsymbol{Q}_o = \begin{bmatrix} \boldsymbol{C} \\ \boldsymbol{C}\boldsymbol{A} \\ \vdots \\ \boldsymbol{C}\boldsymbol{A}^{n-1} \end{bmatrix}$$

Note that this is exactly the same definition as we saw for DT-LTI systems.

Theorem

$\bar{\boldsymbol{x}}$ is an unobservable state if and only if $\bar{\boldsymbol{x}} \in \mathrm{null}(\boldsymbol{Q}_o)$.

Proof...

Observable Systems

Definition

A CT-LTI system is **observable** if no $\bar{\boldsymbol{x}} \neq \mathbf{0}$ is an unobservable state.

In other words, like DT-LTI systems, a CT-LTI system is observable if $\dim(\text{null}(\boldsymbol{Q}_o)) = 0$ or, equivalently, if $\text{rank}(\boldsymbol{Q}_o) = n$.

Some possibly useful facts:

- The set of unobservable states is a subspace of $\mathbb{R}^n$.
- This subspace is invariant under $\boldsymbol{A}$, i.e. if $\bar{\boldsymbol{x}}$ is in the set of unobservable states, then so is $\boldsymbol{A}\bar{\boldsymbol{x}}$ (consequence of the Cayley-Hamilton theorem).

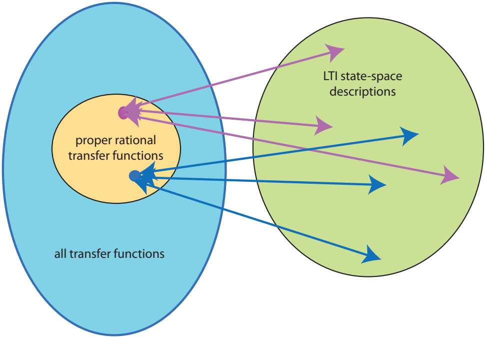
Final Remarks on Reachability, Controllability, and Observability

1. Reachability and Controllability are "control" concepts: "How can we affect the behavior of a system through the input"? This is a preview of Part III of the course.
2. Chapter 6 of Chen has many more results (and examples) than covered in the lecture.
3. Chen does not distinguish between reachability and controllability for CT systems.
   - 3.1 Remember: The set of reachable states and the set of controllable states are the same for CT-LTI systems but not necessarily the same for DT-LTI systems.
   - 3.2 The first mention of reachability is in Section 6.6.1.
4. Please see the discussion of reachability, controllability, and observability for LTV systems in Section 6.8.

Introduction to Realization Theory

Review questions:

1. You are given a p-input, q-output transfer function $\hat{\boldsymbol{G}}(s)$. Can you always find a state-space description for this system?
2. You are given a p-input, q-output, n-state LTI state-space description $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}\}$. Can you always find a transfer function?



Introduction to Realization Theory

Recall that

$$\boxed{\hat{\boldsymbol{G}}(s) = \boldsymbol{C}(s\boldsymbol{I}_n - \boldsymbol{A})^{-1}\boldsymbol{B} + \boldsymbol{D}}$$

- All LTI state-space systems can be converted to transfer functions.
- All transfer functions are not "realizable".
- Any realizable transfer function has infinitely many realizations. For example, if $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}\}$ is a realization, then for any invertible $\boldsymbol{P}$, $\{\boldsymbol{P}^{-1}\boldsymbol{A}\boldsymbol{P}, \boldsymbol{P}^{-1}\boldsymbol{B}, \boldsymbol{C}\boldsymbol{P}, \boldsymbol{D}\}$ is also a realization.
- If a system with n states realizes the transfer function $\hat{\boldsymbol{G}}(s)$, then we can always create a system with more than n states that also realizes $\hat{\boldsymbol{G}}(s)$. Example...

Minimal Realizations

Suppose $\hat{\boldsymbol{G}}(s)$ is realizable. How small can the state dimension n be?

Definition

Given an CT-LTI system with realizable transfer function $\hat{\boldsymbol{G}}(s)$, the **McMillan degree** of the system is the smallest possible state vector dimension over the class of all $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}\}$ that realize the system with transfer function $\hat{\boldsymbol{G}}(s)$.

Definition

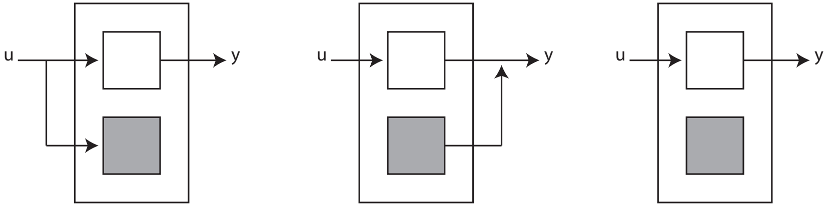
A realization $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}\}$ is **minimal** if its state vector dimension equals the McMillan degree.

Question: Is the minimal realization unique?

Non-Minimal Realizations: Intuition

The system isn't minimal if we have extraneous states.

There are three ways to have extraneous states:



In each case, the states in the shaded box have no impact on the I/O behavior of the system.

Fundamental Theorem of Realization Theory

Theorem

A realization $\{\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}, \boldsymbol{D}\}$ is **minimal** *if and only if $\boldsymbol{A}$ and $\boldsymbol{B}$ are such that the system is reachable and $\boldsymbol{A}$ and $\boldsymbol{C}$ are such that the system is observable.*

This should be intuitively satisfying.

- If a system is reachable, then all states are connected to the input.
- If a system is observable, then all states are connected to the output.

For a system to be minimal, all states must be connected to both the input and the output.

Proof...

Conclusions

Today we covered:

- Reachability for CT systems.
- Controllability for CT systems.
- Observability for CT systems.
- Introduction to realization theory
- Minimal realizations

Next time:

- More realization theory.
- Start Part III of course: Controlling the behavior of systems.