# ECE504 Fall 2009 Final Exam solution

1. $x[1] = Ax[0] + Bu[0]$

$x[2] = Ax[1] + Bu[1] = A^2 x[0] + ABu[0] + Bu[1]$

$= A^2 x[0] + \begin{bmatrix} AB & B \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \end{bmatrix}$

plug in values...

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \end{bmatrix}$$

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} u[0] \\ u[1] \end{bmatrix}$$

A solution is $u[0] = -1$, $u[1] = 0$

This solution is <u>not</u> unique.

2. a) This matrix does not have distinct e-values

$$A - jI_4 = \begin{bmatrix} -j & -1 & 0 & 0 \\ 1 & -j & 0 & 0 \\ 0 & 0 & -j & -1 \\ 0 & 0 & 1 & -j \end{bmatrix} \Rightarrow \begin{bmatrix} j \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ j \\ 1 \end{bmatrix} \text{ are linearly independent e-vectors.}$$

$$A + jI_4 = \begin{bmatrix} j & -1 & 0 & 0 \\ 1 & j & 0 & 0 \\ 0 & 0 & j & -1 \\ 0 & 0 & 1 & j \end{bmatrix} \Rightarrow \begin{bmatrix} -j \\ 1 \\ 0 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} 0 \\ 0 \\ -j \\ 1 \end{bmatrix} \text{ are linearly independant e-vectors.}$$

Hence, since the algebraic multiplicity of each e-value is the same as its geometric multiplicity ($r_1 = m_1 = 2$ and $r_2 = m_2 = 2$) this A matrix is diagonalizable.

b) The TF can be computed by noticing that the 3rd and 4th states are not coupled to the input or the output or the first two states.

The effective system from an input/output perspective is $\bar{A} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ $\bar{B} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\bar{C} = \begin{bmatrix} 1 & 0 \end{bmatrix}$ $\bar{D} = 0$

$$\hat{g}(s) = \bar{C}(sI_2 - \bar{A})^{-1}\bar{B} + \bar{D}$$

$$= [1 \;\; 0]\begin{bmatrix} s & 1 \\ -1 & s \end{bmatrix}^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$\boxed{\hat{g}(s) = \frac{s}{s^2+1}}$$

c) This system is <u>not</u> asymptotically stable because the real part of all e-values is not strictly less that 0.

This system is (internally) stable because $Re(\lambda_i) \le 0$ and all e-values with real part $= 0$ have geometric multiplicity equal to their algebraic mult.

This system is not BIBO stable because the poles of the TF do not have strictly negative real parts

d) $\bar{A}$, $\bar{B}$, $\bar{C}$, and $\bar{D}$ from the previous page is a minimal realization

check: $Q_r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (full rank) $\quad Q_o = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ (full rank) $\quad$ reachable + observable = minimal.

3. First order minors $\left\{\frac{1}{s}, 0\right\}$ $\qquad \hat{G}(s) = \begin{bmatrix} 1/s & 1/s \\ 0 & 1/s \end{bmatrix}$

second order minors : $\left\{\frac{1}{s^2}\right\}$

Least common denominator $= s^2$

$\Rightarrow$ McMillan degree $= 2$

An integrator has a SS realization: $\dot{x}_i(t) = u_p(t)$ ; $y_q(t) = x_i(t)$

Hence, we can realize this system with

$$\dot{\underline{x}}(t) = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix}}_{B} \qquad \underline{y}(t) = \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{C}\underline{x}(t) \qquad \text{with } A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ and } D = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

and $D = 0_{2\times 2}$

This realization is clearly <u>not minimal</u>.

4. a) $Q_r = \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}$ rank = 1 $\Rightarrow$ system is not reachable $\Rightarrow$ e-values can't be placed arbitrarily.

b) $A - BF$ direct method

$$\begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} f_1 & f_2 \end{bmatrix} = \begin{bmatrix} 2-f_1 & 1-f_2 \\ 0 & -2 \end{bmatrix}$$

desired char poly = $(\lambda+1)(\lambda+2) = \lambda^2 + 3\lambda + 2$

actual char poly = $(\lambda - (2-f_1))(\lambda+2)$

$= \lambda^2 + (2-2+f_1)\lambda - 2(2-f_1)$

$\boxed{f_1 = 3 \; ; \; f_2 = \text{anything}}$ $F = \begin{bmatrix} 3 & 0 \end{bmatrix}$ works.

c) Clearly $F$ is not unique

d) The system with state feedback is asymptotically stable (both e-values have negative real part), hence it is also stable and BIBO stable.

e) State feedback doesn't change reachability.

check: $(A - BF) = \begin{bmatrix} -1 & 1 \\ 0 & -2 \end{bmatrix}$

$Q_r = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$ still <u>not reachable</u> system

f) $Q_o = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$ system with feed back is <u>not observable</u> although, if we set $f_2 \neq 0$, it could be observable.

g) $\left.\begin{array}{l} \text{Not reachable} \\ \text{or} \\ \text{Not observable} \\ \text{(or both)} \end{array}\right\} \Rightarrow$ <u>not minimal</u>

5. a) $Q_o$ of <u>uncontrolled</u> system $= \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$ full rank

$\Rightarrow$ we can place e-values arbitrarily.

b) $$A-LC = \begin{bmatrix} 2 & 1 \\ 0 & -2 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ 0 & -2 \end{bmatrix}$$

e-values are at $-1$ and $-2$.

$$(x(t) - w(t)) = e^{t(A-LC)}(x(0) - w(0))$$

e-values at $-1$ and $-2$ $\Rightarrow$ $\|x(t) - w(t)\|^2 \to 0$
as $t \to \infty$ for any mismatch in the initial conditions

c) TF of the controlled system with state feed back and state estimation is the same as the TF of the system with just state feedback.

$$\hat{g}(s) = C(sI - (A-BF))^{-1}B + D$$ $\leftarrow$ system with state feedback.

$$= \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+1 & -1 \\ 0 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 0$$

$$= \frac{1}{(s+1)(s+2)} \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} s+2 & 1 \\ 0 & s+1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$= \frac{s+2}{(s+1)(s+2)} = \boxed{\frac{1}{s+1} = \hat{g}(s)}$$