ECE504 Homework 10 solution

1. $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 1 & 0 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 2 & 0 \end{bmatrix}$

a) first check reachability

$Q_r = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ rank = 3 $\Rightarrow$ reachable.

we can place e-values arbitrarily with state feedback

desired e-values at $-2, -2, -3$.

you can use any of the methods covered in lecture to find $F = \begin{bmatrix} 5 & -5 & 6 \end{bmatrix}$

$A - BF = \begin{bmatrix} -5 & 6 & -6 \\ -5 & 5 & -7 \\ -4 & 5 & -7 \end{bmatrix}$ check e-values = $-3, -2, -2$ ✓

b) closed-loop TF

$$\hat{g}(s) = C(sI - (A-BF))^{-1}B + D = \frac{3s^2 + 2s - 2}{s^3 + 7s^2 + 16s + 12}$$

no pole/zero cancellations.

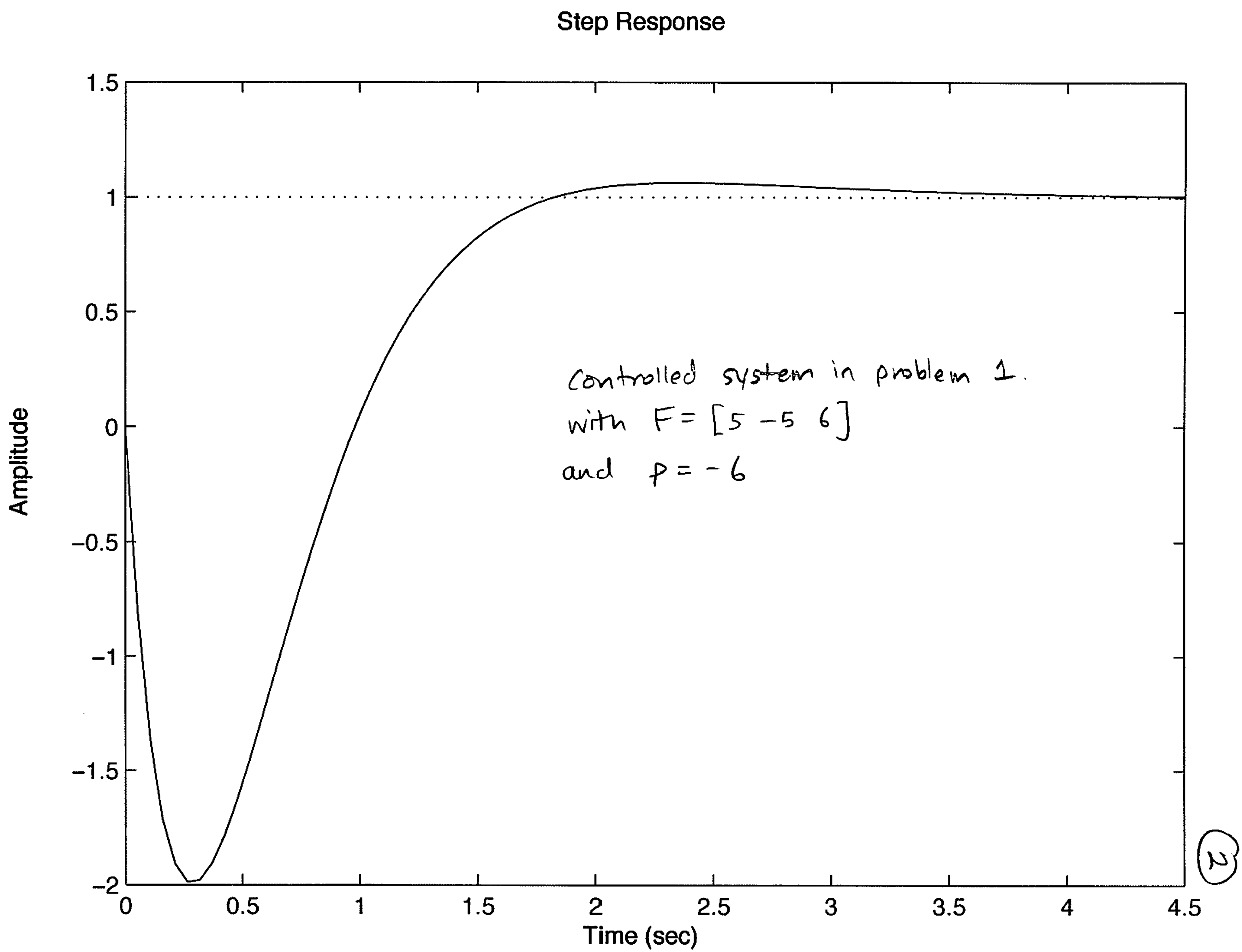
c) We want $p = \frac{\bar{a}_0}{b_0} = \frac{12}{-2} = -6.$

The following page shows the unit step response of the controlled system when $p = -6$. You can see that it settles to the value of $+1$, which is the desired value.

d) The closed loop system is controllable because feedback doesn't affect reachability/controllability.

$Q_o = \begin{bmatrix} C \\ C(A-BF) \\ C(A-BF)^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ -15 & 16 & -20 \\ 75 & -110 & 118 \end{bmatrix}$ rank = 3 $\Rightarrow$ observable

(which we could expect because there were no pole/zero cancellations)


Step Response
Amplitude
Time (sec)
1.5
1
0.5
0
−0.5
−1
−1.5
−2
0
0.5
1
1.5
2
2.5
3
3.5
4
4.5
Controlled system in problem 1.
with F = [5 −5 6]
and p = −6
2

(this space intentionally left blank)

2. a) $\det(\lambda I_2 - (A - LC)) = \begin{vmatrix} \lambda + l_1 & -1 - l_1 \\ l_2 & \lambda - l_2 \end{vmatrix}$
(use direct method)

$$= (\lambda + l_1)(\lambda - l_2) + l_2(1 + l_1)$$
$$= \lambda^2 + (l_1 - l_2)\lambda + l_2$$

If desired e-values are both at $-\alpha$ then desired characteristic polynomial is
$(\lambda+\alpha)(\lambda+\alpha) = \lambda^2 + 2\alpha\lambda + \alpha^2$

hence $l_2 = \alpha^2$ and $l_1 = \alpha^2 + 2\alpha$. $\Rightarrow L = \begin{bmatrix} \alpha^2 + 2\alpha \\ \alpha^2 \end{bmatrix}$

The entries of L are large when α is large.

b) $\dot{x} = Ax + Bu$

$\dot{w} = Aw + Bu + L(y - Cw)$ $\quad (D=0)$

$y = Cx + \epsilon$

$\dot{x} - \dot{w} = Ax - Aw - L(Cx + \epsilon - Cw)$

$\dot{x} - \dot{w} = = (A-LC)(x-w) - L\epsilon$

let $v = x - w$ (The state estimator error) then

$$\underbrace{\dot{v} = (A-LC)v - L\epsilon}$$

This is just an LTI system with $\bar{A} = A - LC$, $\bar{B} = L$, $\bar{u} = -\epsilon$, $\bar{x} = v$

$$\boxed{v(t) = e^{(A-LC)t}v(0) - \int_0^t e^{(A-LC)(t-\tau)} L\epsilon \, d\tau}$$ ← look at $\lim_{t\to\infty} v(t)$

We know that e-values of $A-LC$ are at $-\alpha$ and that α is a positive real number.

$$\Rightarrow \lim_{t\to\infty} e^{(A-LC)t} v(0) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

this gets smaller as $t\to\infty$

what about

$$-\lim_{t\to\infty} \int_0^t e^{(A-LC)(t-\tau)} L\epsilon \, d\tau \quad ?$$

we know $(A-LC)^{-1}$ exists since $\alpha \neq 0$.

$$= (A-LC)^{-1} \lim_{t\to\infty} \left[e^{(A-LC)(t-\tau)} \right]\Bigg|_0^t L\epsilon$$

$$= (A-LC)^{-1} \left[I_2 - 0 \right] L\epsilon = \underline{\underline{(A-LC)^{-1} L\epsilon}}$$

the state estimation error does not go to zero!

Can actually compute it since 2×2 ...

$$(A-LC)^{-1} = \begin{bmatrix} -\alpha^2 - 2\alpha & 1 + \alpha^2 + 2\alpha \\ -\alpha^2 & \alpha^2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & -\frac{1}{\alpha^2} - 1 - \frac{2}{\alpha} \\ 1 & -1 - \frac{2}{\alpha} \end{bmatrix}$$

$$L\epsilon = \begin{bmatrix} (\alpha^2 + 2\alpha)\epsilon \\ \alpha^2 \epsilon \end{bmatrix} \Rightarrow (A-LC)^{-1} L\epsilon = \begin{bmatrix} -\epsilon \\ 0 \end{bmatrix}$$

Surprisingly, the state observer error $\lim_{t\to\infty} v(t) = \begin{bmatrix} -\epsilon \\ 0 \end{bmatrix}$ which is independent of the observer gain.

3. $$x(t) = e^{At}x(0) + \int_0^t e^{A(t-\tau)} Bu(\tau)\,d\tau$$

$$e^{At} = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$ (straight from series defn.)

$$e^{A(t-\tau)} = \begin{bmatrix} 1 & t-\tau \\ 0 & 1 \end{bmatrix}$$

$$x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0) + \int_0^t \begin{bmatrix} 1 & t-\tau \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} u(\tau)\,d\tau$$

$$x(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0) + \begin{bmatrix} \int_0^t u(\tau)\,d\tau \\ 0 \end{bmatrix}$$ <u>true state</u> (1)

now <u>estimated state</u>:

$$w(t) = e^{\bar{A}t} w(0) + \int_0^t e^{\bar{A}(t-\tau)} Bu(\tau)\,d\tau$$

where $\bar{A} = \begin{bmatrix} 0 & 1+\varepsilon \\ 0 & 0 \end{bmatrix}$

same steps...

$$w(t) = \begin{bmatrix} 1 & (1+\varepsilon)t \\ 0 & 1 \end{bmatrix} w(0) + \begin{bmatrix} \int_0^t u(\tau)\,d\tau \\ 0 \end{bmatrix}$$ (2)

use (1) and (2) to compute

$$x(t) - w(t) = \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} x(0) - \begin{bmatrix} 1 & (1+\varepsilon)t \\ 0 & 1 \end{bmatrix} w(0)$$

- Note that $x(t) - w(t)$ is <u>not affected by the input.</u>

$$\begin{bmatrix} x_1(t) - w_1(t) \\ x_2(t) - w_2(t) \end{bmatrix} = \begin{bmatrix} x_1(0) - w_1(0) + t\,(x_2(0) - (1+\varepsilon)\,w_2(0)) \\ x_2(0) - w_2(0) \end{bmatrix}$$

- Note that $x_2(t) - w_2(t)$ is a <u>constant</u> (not a fn. of t)
- Note that $x_1(t) - w_1(t)$ <u>blows up</u> if $x_2(0) \neq (1+\varepsilon) w_2(0)$

Hence a model mismatch ($A \neq \bar{A}$) can cause big problems in state estimators.