ECE 504 : HW #1

1. (a) Applying Newton's second law,

total force acting on the pendulum = mass × acceleration

$$\Rightarrow \quad u(t) - Mg\sin(\theta(t)) = ML\ddot{\theta}(t)$$

Notice that the acceleration is the double derivative of the arc length which is given by $L\theta(t)$

$$\Rightarrow \quad ML\frac{d^2\theta(t)}{dt^2} = u(t) - Mg\sin(\theta(t))$$

$$\Rightarrow \quad \boxed{\frac{d^2\theta(t)}{dt^2} + \frac{g}{L}\sin(\theta(t)) - \frac{1}{ML}u(t) = 0}$$

The above differential equation can also be obtained by equating the torque (product of Moment of Inertia, I and angular acceleration $\ddot{\theta}(t)$) acting on the pendulum and the tangential component of $\vec{L}\times\vec{F}$, which is nothing more than $Lu(t) - MgL\sin(\theta(t))$.

(b) (i) Lumped, because the input/output is a function of time alone and does not depend on any other independent variable. So, an ordinary differential equation is sufficient to describe the system's behavior. If the system was distributed, a partial differential equation is needed.

(ii) Causal, because output at ~~the~~ any time instant depends on input at the current time instant.

(iii) Non linear because $\sin(\ )$ of the output appears in the differential equation.

(iv) Time-invariant, because the coefficients that appear in the equation are not functions of time.

(c) Assuming $\theta(t)$ to be small, $\sin(\theta(t)) \approx \theta(t)$.

The linear approximation of the differential equation is

$$\boxed{\frac{d^2\theta(t)}{dt^2} + \frac{g}{L}\theta(t) - \frac{1}{ML}u(t) = 0}$$

(d) Transfer function:

Taking Laplace transform of the above equation,

$$s^2\Theta(s) - s\theta(0) - \dot{\theta}(0) + \frac{g}{L}\Theta(s) - \frac{1}{ML}U(s) = 0$$

$$\Rightarrow \left(s^2 + \frac{g}{L}\right)\Theta(s) - s\theta(0) - \dot{\theta}(0) = \frac{1}{ML}U(s)$$

$$\Rightarrow \frac{\Theta(s)}{U(s)} = \frac{1/ML}{s^2 + \frac{g}{L}}$$ (ignoring the initial conditions)

Impulse response: If $u(t) = \delta(t)$, then $U(s) = 1$

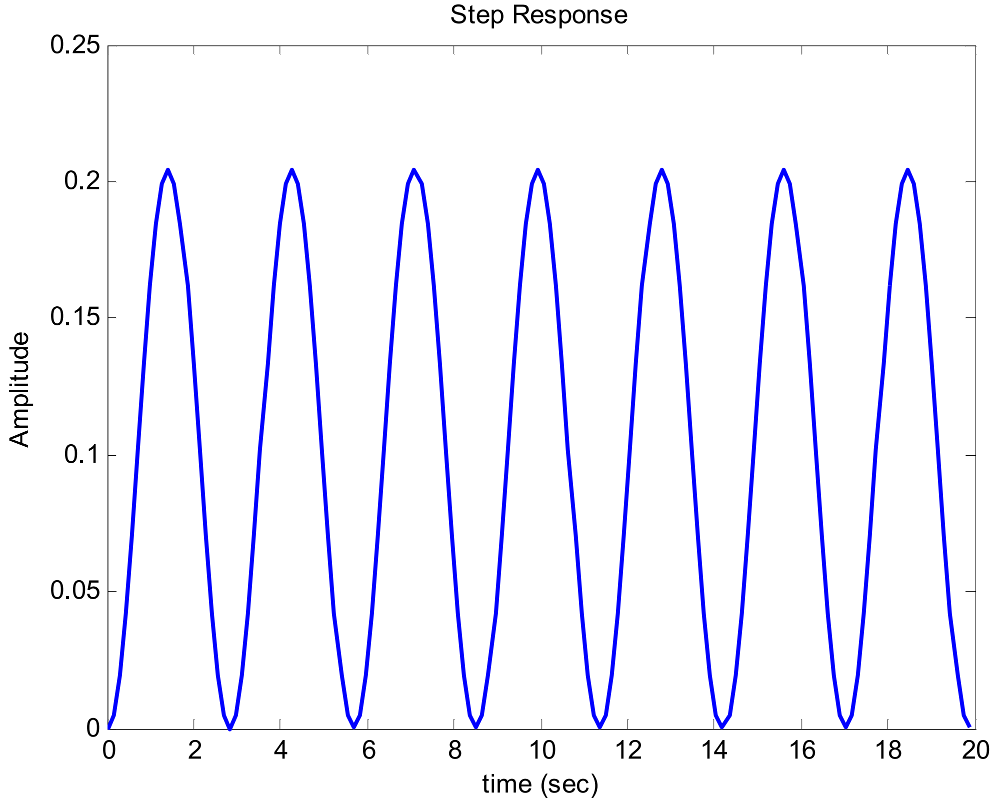
$$\Rightarrow \Theta(s) = \frac{1/ML}{s^2 + \frac{g}{L}}$$

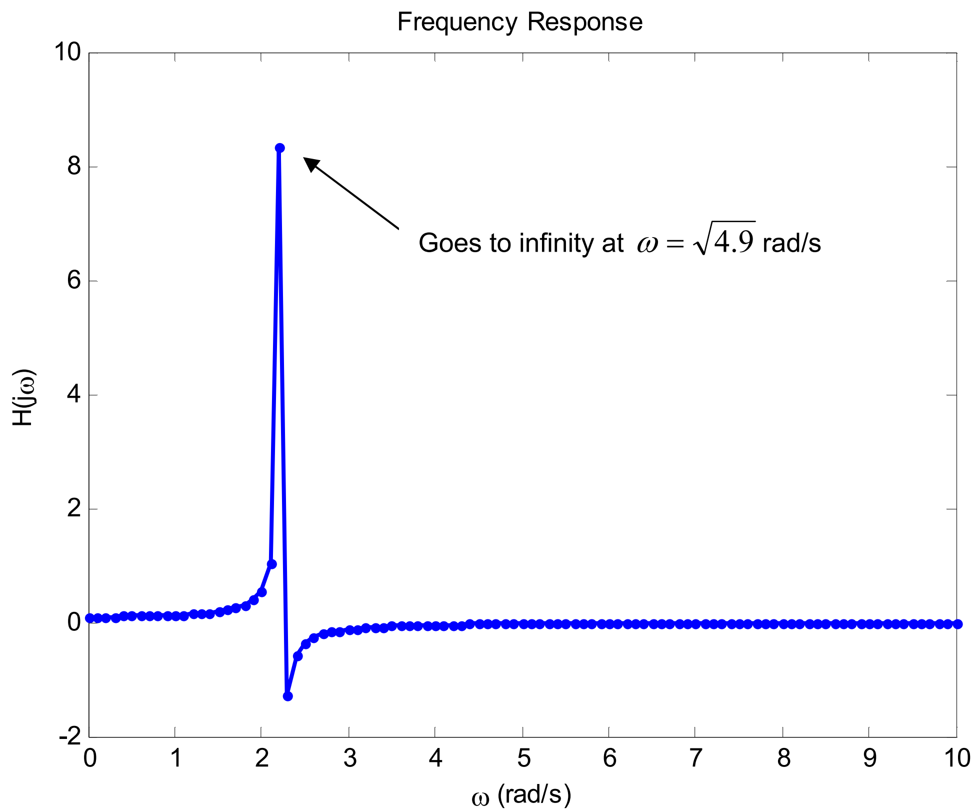
let $\sqrt{\frac{g}{L}} = \alpha \Rightarrow \Theta(s) = \frac{1}{M\sqrt{gL}} \frac{\alpha}{s^2+\alpha^2}$

Applying inverse laplace transform,

$$\theta(t) = \frac{1}{M\sqrt{gL}} \sin(\alpha t)\, U(t)$$

Solution for Problem 1(e)





Solution for Problem 1(e)

```
%MATLAB script to find the step and
%frequency responses of the system in 1(c)

clc;
clear; close all;

M=1;L=2;g=9.8;

b=[1/(M*L)]; %Numerator of the transfer function
a=[1 0 g/L]; %Denominator of the transfer function
[A,B,C,D]=tf2ss(b,a); %convert transfer function filter
                      %parameters to state-space form

sys=ss(A,B,C,D); %creates a state-space object representing
                 %the continuous-time state-space model

figure(1);
[y,t]=step(sys,20);
plot(t,y);
xlabel('time (sec)');
ylabel('Amplitude');
title('Step Response');

w=0:0.1:10;
H=freqresp(sys,w);
H=squeeze(H);
figure(2);
plot(w,H,'.-');
xlabel('\omega (rad/s)');
ylabel('H(j\omega)');
title('Frequency Response');
```

ECE504 Fall 2009 Solution to HW1

Chen 2.3

- This system is <u>linear</u> because it satisfies the additivity and homogeneity properties.

  It is memoryless, so we don't need to worry about the state

  If we apply $u_1(t) + u_2(t)$, we get

  $$y(t) = \begin{cases} u_1(t) + u_2(t) & \text{for } t \le \alpha \\ 0 & \text{for } t > \alpha \end{cases}$$

  which confirms additivity. If we apply $\beta u(t)$, we get

  $$y(t) = \begin{cases} \beta u(t) & \text{for } t \le \alpha \\ 0 & \text{for } t > \alpha \end{cases}$$

  which confirms homogeneity.

- This system is causal because it is memoryless.

- This system is time varying. This can be seen by example. Suppose $\alpha = 1$ and we apply $u(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$ i.e., a unit step function

  Then $y(t) = \begin{cases} u(t) & \text{for } t \le 1 \\ 0 & \text{for } t > 1 \end{cases}$

  $$= \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

  Now suppose we apply a delayed unit step function

  $$u(t) = \begin{cases} 1 & t \ge 2 \\ 0 & t < 2 \end{cases}$$

  Then $y(t) = \begin{cases} u(t) & \text{for } t \le 1 \\ 0 & \text{for } t > 1 \end{cases}$

  $$= 0 \ne y(t-2) \Rightarrow \text{time varying}$$

Chen 2.10

Recall that the transfer function implicitly assumes relaxed initial conditions.

$$(s^2+2s-3)\hat{y}(s) = (s-1)\hat{u}(s)$$

$$\Rightarrow \hat{g}(s) = \frac{\hat{y}(s)}{\hat{u}(s)} = \frac{s-1}{(s+3)(s-1)} = \frac{1}{s+3}$$

The impulse response can be computed by taking the inverse Laplace transform...

$$g(t) = \mathcal{L}^{-1}\{\hat{g}(s)\} = \begin{cases} e^{-3t} & t \geq 0 \\ 0 & t < 0 \end{cases}$$

causal, linear, lumped, time-invariant system