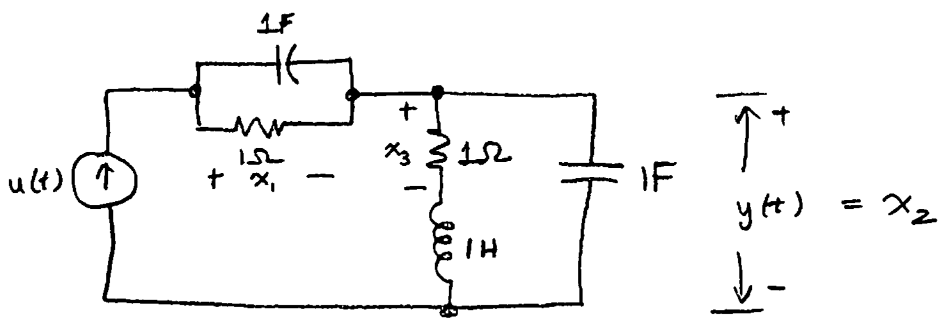
ECE504 Fall 2009 Solution to HW2

Chen 2.19



- Let x_1 be voltage across parallell resistor/capacitor at top.

  then KCL says $u(t) = C\dfrac{dx_1}{dt} + \dfrac{x_1}{R} = \dot{x}_1 + x_1$ (1)

- Let x_2 be the voltage at the output

- Let x_3 the The voltage across the other $1\,\Omega$ resistor in series with the inductor

  Then KCL says $\dfrac{x_3}{R} + C\dfrac{dx_2}{dt} = u(t) = x_3 + \dot{x}_2$ (2)

  and KVL says $\dfrac{L}{R}\dfrac{dx_3}{dt} + x_3 = x_2$

$$\dot{x}_3 + x_3 = x_2 \quad (3)$$

So

$$\underline{\dot{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix}}_{A} \underline{x} + \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{B} u \quad \Bigg\} \text{ from equations (1), (2), and (3)}$$

and

$$y = \underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{C} x + \underbrace{0}_{D} u \quad \Big\} \text{ from the fact that } y = x_2.$$

continued..

Chen 2.19 continued...

The transfer function can be found by computing

$$C(sI-A)^{-1}B + D = \hat{g}(s)$$

$$sI-A = \begin{bmatrix} s+1 & 0 & 0 \\ 0 & s & +1 \\ 0 & -1 & s+1 \end{bmatrix}$$

$$\det(sI-A) = (s+1)\big(s(s+1)+1\big) = s^3+2s^2+2s+1$$

we don't need to do the whole adjoint since the C vector will select only the middle row of $(sI-A)^{-1}$ and the B vector only selects the first 2 columns of $(sI-A)^{-1}$.

$$(sI-A)^{-1} = \frac{\begin{bmatrix} - & - & - \\ a & b & - \\ - & - & - \end{bmatrix}}{\det(sI-A)}$$

only need these.

$$a = C_{12} \text{ cofactor} = (-1)^3 \det\left(\begin{bmatrix} 0 & +1 \\ 0 & s+1 \end{bmatrix}\right) = 0$$

$$b = C_{22} \text{ cofactor} = (-1)^4 \det\left(\begin{bmatrix} s+1 & 0 \\ 0 & s+1 \end{bmatrix}\right) = (s+1)^2$$

so $$C(sI-A)^{-1}B + D = \frac{(s+1)^2}{s^3+2s^2+2s+1} = \frac{(s+1)^{\cancel{2}}}{\cancel{(s+1)}(s^2+s+1)}$$

$$\boxed{\hat{g}(s) = \frac{s+1}{s^2+s+1}}$$

you can check this result in Matlab with ss2tf

2. $$y[k] = \sum_{n=0}^{N-1} \lambda^n u[k-n]$$

a) $N=3 \Rightarrow y[k] = u[k] + \lambda u[k-1] + \lambda^2 u[k-2]$

$$x[k] = \begin{bmatrix} u[k-1] \\ u[k-2] \end{bmatrix} \qquad x[k+1] = \begin{bmatrix} u[k] \\ u[k-1] \end{bmatrix}$$

$$x[k+1] = \underbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}_{A} x[k] + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u[k]$$

$$y[k] = \underbrace{\begin{bmatrix} \lambda & \lambda^2 \end{bmatrix}}_{C} x[k] + \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{D} u[k]$$

These matrix equations can be written directly from the definition of $x[k]$ and $y[k]$.

b) $N=3 \Rightarrow y[k] = u[k] + \lambda u[k-1] + \lambda^2 u[k-2]$

$$x[k] = \begin{bmatrix} u[k-1] + u[k-2] \\ u[k-1] \end{bmatrix}$$

$$x[k+1] = \begin{bmatrix} u[k] + u[k-1] \\ u[k] \end{bmatrix}$$

From these equations, we can directly write

$$x[k+1] = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{A} x[k] + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{B} u[k]$$

$$y[k] = \underbrace{\begin{bmatrix} \lambda^2 & \lambda - \lambda^2 \end{bmatrix}}_{C} x[k] + \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{D} u[k]$$

continued...

c) In parts a and b, we chose different states for the same system, i.e. the system has the same input/output response even though A, B, C, and D are different. Hence, we can say that there are lots of state space descriptions that give the same input-output response

d) $y[k] = \sum_{n=0}^{N-1} \lambda^n u[k-n] - y[k-1]$

$N=3 \Rightarrow y[k] = u[k] + \lambda u[k-1] + \lambda^2 u[k-2] - y[k-1]$

write this as a transfer function:

$(1+z^{-1})Y(z) = (1+\lambda z^{-1} + \lambda^2 z^{-2})U(z)$ + initial conditions

TF assumes relaxed initial conditions, hence

$$\hat{g}(z) = \frac{Y(z)}{U(z)} = \frac{1+\lambda z^{-1} + \lambda^2 z^{-2}}{1+z^{-1}}$$

multiply by $\frac{z^2}{z^2}$ to get $\frac{N(z)}{D(z)} = \frac{z^2+\lambda z+\lambda^2}{z^2+z+0}$

One way to get a SS description is to use the "trick" given in lecture:

$$\hat{v}(z) = \frac{1}{D(z)}\hat{u}(z) \Leftrightarrow D(z)\hat{v}(z) = \hat{u}(z)$$

$$\Leftrightarrow \underbrace{v[k+2] + v[k+1] + 0v[k] = u[k]}_{v[k+2] = u[k] - v[k+1]}$$

let $x[k] = \begin{bmatrix} v[k+1] \\ v[k] \end{bmatrix}$

then $x[k+1] = \begin{bmatrix} v[k+2] \\ v[k+1] \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ 1 & 0 \end{bmatrix}}_{A} x[k] + \underbrace{\begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{B} u[k]$

continued ...

now $\hat{y}(z) = N(z)\hat{v}(z) = (z^2 + \lambda z + \lambda^2)\hat{v}(z)$

$$\Leftrightarrow \quad y[k] = v[k+2] + \lambda v[k+1] + \lambda^2 v[k]$$

recall that $v[k+2] = u[k] - v[k+1]$

so

$$y[k] = u[k] + (\lambda - 1)\, v[k+1] + \lambda^2 v[k]$$

$$y[k] = \underbrace{\begin{bmatrix} \lambda - 1 & \lambda^2 \end{bmatrix}}_{C} + \underbrace{\begin{bmatrix} 1 \end{bmatrix}}_{D} u[k]$$

Note the state here is also dimension 2, even though we added output feedback.

check answer:

$$\hat{g}(z) = C(zI - A)^{-1}B + D = \begin{bmatrix} \lambda-1 & \lambda^2 \end{bmatrix} \begin{bmatrix} z+1 & 0 \\ -1 & z \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1$$

$$= \left(\frac{1}{z(z+1)}\right) \begin{bmatrix} \lambda-1 & \lambda^2 \end{bmatrix} \begin{bmatrix} z & 0 \\ 1 & z+1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 1$$

$$= \frac{(\lambda-1)z + \lambda^2}{z(z+1)} + 1 = \frac{z(z+1) + (\lambda-1)z + \lambda^2}{z^2 + z}$$

$$= \frac{z^2 + \cancel{z} + \lambda z - \cancel{z} + \lambda^2}{z^2 + z} = \frac{z^2 + \lambda z + \lambda^2}{z^2 + z} \quad \checkmark$$

3.

$$W = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{bmatrix}$$

use adjoint/determinant method to compute W^{-1}.

$$\det(W) = a \det\left(\begin{bmatrix} b & c \\ d & e \end{bmatrix}\right) = a(be - cd)$$

$$\operatorname{adj}(W) = \begin{bmatrix} C_{11} & C_{12} & C_{13} \\ C_{21} & C_{22} & C_{23} \\ C_{31} & C_{32} & C_{33} \end{bmatrix}^T$$

$$C_{11} = (-1)^{1+1} \det\left(\begin{bmatrix} b & c \\ d & e \end{bmatrix}\right) = be - cd$$

$$C_{12} = (-1)^{1+2} \det\left(\begin{bmatrix} 0 & c \\ 0 & e \end{bmatrix}\right) = 0$$

$C_{13} = 0$ (same reasons)

$C_{21} = 0$ "

$$C_{22} = (-1)^{2+2} \det\left(\begin{bmatrix} a & 0 \\ 0 & e \end{bmatrix}\right) = ae$$

$$C_{23} = (-1)^{2+3} \det\left(\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}\right) = -ad$$

$C_{31} = 0$

$$C_{32} = (-1)^{2+3} \det\left(\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}\right) = -ac$$

$$C_{33} = (-1)^{3+3} \det\left(\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix}\right) = ab$$

hence

$$W^{-1} = \begin{bmatrix} \frac{1}{a} & 0 & 0 \\ 0 & \frac{e}{be-cd} & \frac{-c}{be-cd} \\ 0 & \frac{-d}{be-cd} & \frac{b}{be-cd} \end{bmatrix}$$

you can check this with random numbers in Matlab

The inverse exists when $\det(W) \neq 0$. Hence, we need $a \neq 0$ and $be \neq cd$ for W^{-1} to exist.