ECE504 HW3 Solution

1. $$\hat{g}(z) = \frac{z^{-2}+5z^{-1}-4}{z^{-2}+2z^{-1}-3} = \frac{1+5z-4z^2}{1+2z-3z^2} = \frac{N(z)}{D(z)}$$

SS1: $\hat{v}(z) = \frac{1}{D(z)}\hat{u}(z)$

$$(1+2z-3z^2)\,\hat{v}(z) = \hat{u}(z)$$

$$v[k] + 2v[k+1] - 3v[k+2] = u[k] \quad \Rightarrow v[k+2] = \frac{v[k]+2v[k+1]-u[k]}{3}$$

let $\underline{x}[k] = \begin{bmatrix} v[k+1] \\ v[k] \end{bmatrix}$

Then $$\boxed{\underline{x}[k+1] = \begin{bmatrix} 2/3 & 1/3 \\ 1 & 0 \end{bmatrix} \underline{x}[k] + \begin{bmatrix} -1/3 \\ 0 \end{bmatrix} u[k]}$$

$$\hat{y}(z) = N(z)\hat{v}(z) = (1+5z-4z^2)\,\hat{v}(z)$$

$$\begin{aligned} y[k] &= v[k] + 5v[k+1] - 4v[k+2] \\ &= v[k] + 5v[k+1] - 4\left(\frac{v[k]+2v[k+1]-u[k]}{3}\right) \\ &= -\frac{1}{3}v[k] + \frac{7}{3}v[k+1] + \frac{4}{3}u[k] \end{aligned}$$

$$\boxed{y[k] = \begin{bmatrix} 7/3 & -1/3 \end{bmatrix} \underline{x}[k] + \frac{4}{3}u[k]}$$

check: $$C(zI-A)^{-1}B + D = \begin{bmatrix} 7/3 & -1/3 \end{bmatrix} \begin{bmatrix} z-2/3 & -1/3 \\ -1 & z \end{bmatrix}^{-1} \begin{bmatrix} -1/3 \\ 0 \end{bmatrix} + 4/3$$

$$= \frac{-4z^2+5z+1}{-3z^2+2z+1} \quad \checkmark$$

continued...

SS2 : Lets redefine the state. (This is only one way to get a second SS. representation)

let $\underline{x}[k] = \begin{bmatrix} v[k+1] + v[k] \\ v[k] \end{bmatrix}$ then

$$\boxed{\begin{aligned} \underline{x}[k+1] &= \begin{bmatrix} 5/3 & -4/3 \\ 1 & -1 \end{bmatrix} \underline{x}[k] + \begin{bmatrix} -1/3 \\ 0 \end{bmatrix} u[k] \\ y[k] &= \begin{bmatrix} 7/3 & -8/3 \end{bmatrix} \underline{x}[k] + \frac{4}{3} u[k] \end{aligned}}$$

first line of state dynamic equation.

$$v[k+2] + v[k+1] = \frac{v[k] + 2v[k+1] - u[k]}{3} + v[k+1] = \frac{1}{3}v[k] + \frac{5}{3}v[k+1] - \frac{1}{3}u[k]$$

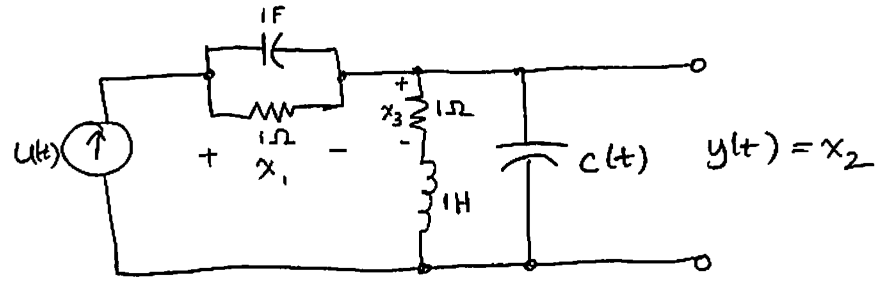
$$= \frac{5}{3}x_1 - \frac{4}{3}x_2 - \frac{1}{3}u$$

second line is straightforward

check: $c(sI - A)^{-1}B + D = \begin{bmatrix} 7/3 & -8/3 \end{bmatrix} \begin{bmatrix} z - 5/3 & 4/3 \\ -1 & z+1 \end{bmatrix}^{-1} \begin{bmatrix} -1/3 \\ 0 \end{bmatrix} + 4/3$

$$= \frac{-4z^2 + 5z + 1}{-3z^2 + 2z + 1} \checkmark$$

2.



same states as HW2

$$u(t) = \dot{x}_1 + x_1 \qquad (1)$$

Note that current through a time-varying capacitor is

$$i = C\frac{dv}{dt} + v\frac{dC}{dt}$$

so KCL says

$$\frac{x_3}{R} + C(t)\frac{dx_2}{dt} + x_2\frac{dC(t)}{dt} = u(t)$$

$$\Rightarrow u(t) = x_3 + c(t)\,\dot{x}_2 + \dot{c}(t)\,x_2 \qquad (2)$$

and KVL says (see HW2 solution)

$$\dot{x}_3 + x_3 = x_2 \qquad (3)$$

so

$$\underline{\dot{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -\frac{\dot{c}(t)}{c(t)} & \frac{1}{c(t)} \\ 0 & 1 & -1 \end{bmatrix}}_{A(t)} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 \\ \frac{1}{c(t)} \\ 0 \end{bmatrix}}_{B(t)} u(t)$$

$$y = \underbrace{\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}}_{c(t)} \underline{x} + \underbrace{0}_{D(t)} u$$

↑ (not the capacitance but the SS matrix)

3.

a) To confirm that this is a nominal solution, just compute the derivatives.

$$\dot{\tilde{x}}_3(t) = \frac{d}{dt}\left(m_0 + u_0 t\right) = u_0 \quad \checkmark \quad \text{since } u(t) \equiv u_0$$

$$\dot{\tilde{x}}_2(t) = \frac{d}{dt}\left(at + b\ln\left(1+\frac{u_0 t}{m_0}\right)\right)$$

$$= a + b\,\frac{1}{1+\frac{u_0 t}{m_0}}\cdot\frac{u_0}{m_0} \qquad \text{(chain rule)}$$

$$= a + b\,\frac{u_0}{m_0 + u_0 t}$$

$$= a + b\,\frac{u_0}{\tilde{x}_3(t)} \quad \checkmark \qquad \text{since } \tilde{x}_3(t) = m_0 + u_0 t \text{ and } u(t) \equiv u_0$$

$$\dot{\tilde{x}}_1(t) = \frac{d}{dt}\left(a\frac{t^2}{2} + b\frac{m_0}{u_0}\left[\left(1+\frac{u_0 t}{m_0}\right)\ln\left(1+\frac{u_0 t}{m_0}\right) - \frac{u_0 t}{m_0}\right]\right)$$

$$= at + b\frac{m_0}{u_0}\left(\frac{u_0}{m_0}\ln\left(1+\frac{u_0 t}{m_0}\right) + \left(1+\frac{u_0 t}{m_0}\right)\frac{1}{1+\frac{u_0 t}{m_0}}\,\frac{u_0}{m_0}\right) - b\frac{m_0}{u_0}\left(\frac{u_0}{m_0}\right)$$

$$= at + b\frac{m_0}{u_0}\left(\frac{u_0}{m_0}\ln\left(1+\frac{u_0 t}{m_0}\right) + \frac{u_0}{m_0}\right) - b$$

$$= at + b\ln\left(1+\frac{u_0 t}{m_0}\right) + b - b$$

$$= \tilde{x}_2(t) \quad \checkmark \qquad \text{since } \tilde{x}_2(t) = at + b\ln\left(1+\frac{u_0}{m_0}t\right)$$

b) Linearize around nominal solution...

$$A(t) = \left.\frac{\partial f}{\partial x}\right|_{\substack{x=\tilde{x}\\ u=\tilde{u}}} = \left.\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -b\,u(t)/x_3^2(t) \\ 0 & 0 & 0 \end{bmatrix}\right|_{\substack{x=\tilde{x}\\ u=\tilde{u}}} = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{-b u_0}{(m_0 + u_0 t)^2} \\ 0 & 0 & 0 \end{bmatrix}}_{\text{note time varying (but linear)}}$$

$$B(t) = \left.\frac{\partial f}{\partial u}\right|_{\substack{x=\tilde{x}\\ u=\tilde{u}}} = \left.\begin{bmatrix} 0 \\ b/x_3 \\ 1 \end{bmatrix}\right|_{\substack{x=\tilde{x}\\ u=\hat{u}}} = \underbrace{\begin{bmatrix} 0 \\ \frac{b}{m_0 + u_0 t} \\ 1 \end{bmatrix}}_{\text{also time varying}}$$

continued...

no need to take derivatives of output function since it is already linear

$$\text{So} \quad \dot{\underline{\tilde{x}}}(t) = \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & \frac{-bu_0}{(m_0+u_0t)^2} \\ 0 & 0 & 0 \end{bmatrix}}_{A(t)} \underline{\tilde{x}}(t) + \underbrace{\begin{bmatrix} 0 \\ \frac{b}{m_0+u_0t} \\ 1 \end{bmatrix}}_{B(t)} u(t)$$

$$\dot{\tilde{y}}(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \dot{\tilde{x}}(t) + 0\,u(t)$$