ECE504 Homework #4 solution

1. a) $A = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ time-invariant

zero input response $x[k] = A^k x[0]$; $y[k] = [1 \ 1 \ 1] x[k]$

$A^0 = I_3$

A^1 = above

$A^2 = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & -2 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$A^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

hence, the zero input response of this system can easily be computed for all 3 cases:

$$\boxed{\begin{aligned} x[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &\Rightarrow x[k] = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ldots \right\} \text{ for } k=0,1,2,\ldots \\ x[0] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} &\Rightarrow x[k] = \left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ldots \right\} \text{ for } k=0,1,2,3,\ldots \\ x[0] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} &\Rightarrow x[k] = \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \ldots \right\} \text{ for } k=0,1,2,3,4,\ldots \end{aligned}}$$

From the output equation, we can then write

$$\boxed{\begin{aligned} x[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} &\Rightarrow y[k] = \{1, 0, 0, \ldots\} \\ x[0] = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} &\Rightarrow y[k] = \{1, 1, 0, 0, \ldots\} \\ x[0] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} &\Rightarrow y[k] = \{1, -1, 1, 0, 0, \ldots\} \end{aligned}}$$

b) Given $x[0] = \begin{bmatrix} \gamma_1 \\ \gamma_2 \\ \gamma_3 \end{bmatrix} = \gamma_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \gamma_2 \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + \gamma_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ ← We have the zero-input responses for each of these 3 initial conditions already.

We can use the <u>linearity</u> of the system to write the general zero-input response as $\boxed{y[k] = \{\gamma_1 + \gamma_2 + \gamma_3, \ \gamma_2 - \gamma_3, \ \gamma_3, \ 0, 0, \ldots\}}$ for $k = 0, 1, 2, 3, 4, \ldots$

2. Chen 4.1

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x(t) = Ax(t) \qquad \underline{\text{LTI}}$$

We can find the STM using the P-B series or the fundamental matrix method. Let's use the latter.

Let $\underline{x}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $x_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ (linearly indep)

The A matrix implies that $\dot{x}_1(t) = x_2(t)$ and $\dot{x}_2(t) = -x_1(t)$

The solutions to these DE's are:

$$\left.\begin{aligned} x_1(t) &= ae^{jt} + be^{-jt} \\ x_2(t) &= aje^{jt} - bje^{-jt} \end{aligned}\right\} \text{ where } a \text{ and } b \text{ are chosen to satisfy the initial conditions}$$

check. $\dot{x}_1(t) = ajе^{jt} - bje^{-jt} = x_2(t)$ ✓

$\dot{x}_2(t) = -ae^{jt} - be^{-jt} = -x_1(t)$ ✓

so, for $x_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, we have $\left.\begin{aligned} a+b&=1 \\ (a-b)j&=0 \end{aligned}\right\} \Rightarrow a = b = \frac{1}{2}$

for $x_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we have $\left.\begin{aligned} a+b&=0 \\ (a-b)j&=1 \end{aligned}\right\} \Rightarrow \begin{aligned} a&=\frac{1}{2j} \\ b&=-\frac{1}{2j} \end{aligned}$

So the fundamental matrix is

$$\underline{\underline{X}}(t) = \begin{bmatrix} \frac{1}{2}(e^{jt}+e^{-jt}) & \frac{1}{2j}(e^{jt}-e^{-jt}) \\ \frac{j}{2}(e^{jt}-e^{-jt}) & \frac{1}{2}(e^{jt}+e^{-jt}) \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

And the STM from time $t_0 = 0$ is

$$\Phi(t,0) = \underline{\underline{X}}(t)\, X^{-1}(0) \qquad \text{where } \underline{\underline{X}}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

hence $\underline{\underline{X}}^{-1}(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Hence $\boxed{\Phi(t,0) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}}$

and $x(t) = \Phi(t,0)\, x(0)$. This is what we wanted to show.

3. $A(t) = \begin{bmatrix} 0 & e^{-t} \\ 0 & 1 \end{bmatrix}$; find the STM $\Phi(t,s)$

$\dot{x}_2(t) = x_2(t) \Rightarrow x_2(t) = a e^{t}$ (a is chosen to satisfy the initial condition)

$\dot{x}_1(t) = e^{-t} x_2(t) = a \Rightarrow x_1(t) = at + b$ (b is chosen to satisfy the initial cond.)

fundamental matrix method again...

$\underline{x}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow b = 1,\ a = 0$

$\underline{x}_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow b = 0,\ a = 1$

Hence $\underline{\underline{X}}(t) = \begin{bmatrix} 1 & t \\ 0 & e^{t} \end{bmatrix}$ if $\underline{\underline{X}} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then $\underline{\underline{X}}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

$\underline{\underline{X}}^{-1}(s) = \begin{bmatrix} e^{s} & -s \\ 0 & 1 \end{bmatrix} \cdot \frac{1}{e^{s}} = \begin{bmatrix} 1 & -se^{-s} \\ 0 & e^{-s} \end{bmatrix}$

$$\boxed{\Phi(t,s) = \underline{\underline{X}}(t)\,\underline{\underline{X}}^{-1}(s) = \begin{bmatrix} 1 & t \\ 0 & e^{t} \end{bmatrix}\begin{bmatrix} 1 & -se^{-s} \\ 0 & e^{-s} \end{bmatrix} = \begin{bmatrix} 1 & e^{-s}(t-s) \\ 0 & e^{t-s} \end{bmatrix}}$$

Check: $\Phi(t,t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ ✓

$$\frac{d}{dt}\Phi(t,s) = \begin{bmatrix} 0 & e^{-s} \\ 0 & e^{t-s} \end{bmatrix} \stackrel{?}{=} \underbrace{\begin{bmatrix} 0 & e^{-t} \\ 0 & 1 \end{bmatrix}}_{A(t)} \underbrace{\begin{bmatrix} 1 & e^{-s}(t-s) \\ 0 & e^{t-s} \end{bmatrix}}_{\Phi(t,s)} = \begin{bmatrix} 0 & e^{-s} \\ 0 & e^{t-s} \end{bmatrix}$$ ✓

What if we tried the Peano-Baker series?

$$M_k(t) = \begin{cases} I_n & k = 0 \\ \int_s^t \int_s^{\tau_1} \cdots \int_s^{\tau_{k-1}} A(\tau_1) A(\tau_2) \cdots A(\tau_k)\, d\tau_k \cdots d\tau_1 \end{cases}$$

Note that $A(\tau_1) \cdots A(\tau_k) = \begin{bmatrix} 0 & e^{-\tau_1} \\ 0 & 1 \end{bmatrix} \cdots \begin{bmatrix} 0 & e^{\tau_k} \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & e^{-\tau_1} \\ 0 & 1 \end{bmatrix} = A(\tau_1)$

so $M_k(t) = \int_s^t A(\tau_1)\left[\int_s^{\tau_1} \cdots \int_s^{\tau_{k-1}} d\tau_k \cdots d\tau_2\right] d\tau_1 = \int_s^t A(\tau_1)\left[\frac{(\tau_1 - s)^{k-1}}{(k-1)!}\right] d\tau_1$ for $k \geq 1$

$$= \int_s^t \begin{bmatrix} 0 & e^{-\tau_1}\frac{(\tau_1 - s)^{k-1}}{(k-1)!} \\ 0 & \frac{(\tau_1 - s)^{k-1}}{(k-1)!} \end{bmatrix} d\tau_1 = \begin{bmatrix} 0 & ? \\ 0 & \frac{(t-s)^k}{k!} \end{bmatrix}$$

continued...

3. continued...

Let's look at the upper right term for the first few elements of the series

k	upper right term of $M_k(t)$
0	0
1	$\int_s^t e^{-\tau_1}\,d\tau_1 = e^{-s} - e^{-t}$
2	$\int_s^t e^{-\tau_1}(\tau_1 - s)\,d\tau_1 = e^{-s} - e^{-t} - (t-s)e^{-t}$
3	$\int_s^t e^{-\tau_1}\frac{(\tau_1 - s)^2}{2}\,d\tau_1 = e^{-s} - e^{-t} - (t-s)e^{-t} - \frac{(t-s)^2}{2}e^{-t}$
$\vdots$	$\vdots$

So upper right term of $M_k(t) = e^{-s} - e^{-t}\sum_{\ell=0}^{k-1}\frac{(t-s)^\ell}{\ell!} = e^{-s}\left[1 - e^{-(t-s)}\sum_{\ell=0}^{k-1}\frac{(t-s)^\ell}{\ell!}\right]$

$$= e^{-s}\left\{1 - e^{-(t-s)}\left[e^{(t-s)} - \sum_{\ell=k}^{\infty}\frac{(t-s)^\ell}{\ell!}\right]\right\}$$

$$= e^{-s}\left\{1 - 1 + e^{-(t-s)}\sum_{\ell=k}^{\infty}\frac{(t-s)^\ell}{\ell!}\right\} = e^{-s}\left(e^{-(t-s)}\sum_{\ell=k}^{\infty}\frac{(t-s)^\ell}{\ell!}\right)$$

To compute $\Phi(t,s) = \sum_{k=0}^{\infty} M_k(t)$

$$= \begin{bmatrix} 1 & 0 + e^{-s}\overbrace{\left[\sum_{k=1}^{\infty} e^{-(t-s)}\sum_{\ell=k}^{\infty}\frac{(t-s)^\ell}{\ell!}\right]}^{\text{what is this?}} \\ 0 & 1 + \sum_{k=1}^{\infty}\frac{(t-s)^k}{k!} \end{bmatrix}$$

$= e^{t-s}$ ✓

continued...

· problem 3 continued...

$$\sum_{k=1}^{\infty} e^{-(t-s)} \sum_{l=k}^{\infty} \frac{(t-s)^l}{l!} = e^{-(t-s)} \sum_{l=1}^{\infty} \frac{(t-s)^l}{l!} \qquad k=1$$

$$+ e^{-(t-s)} \sum_{l=2}^{\infty} \frac{(t-s)^l}{l!} \qquad k=2$$

$$+ e^{-(t-s)} \sum_{l=3}^{\infty} \frac{(t-s)^l}{l!} \qquad k=3$$

$$+ \dots , \text{ etc.}$$

Note that

$l=1$ appears only once

$l=2$ appears twice

$l=3$ appears thrice, etc...

rewrite:

$$= e^{-(t-s)} \sum_{l=1}^{\infty} l \frac{(t-s)^l}{l!} = e^{-(t-s)} \sum_{l=1}^{\infty} \frac{(t-s)^l}{(l-1)!}$$

$$= e^{-(t-s)} (t-s) \sum_{l=1}^{\infty} \frac{(t-s)^{l-1}}{(l-1)!}$$

$$= e^{-(t-s)} (t-s) \underbrace{\sum_{m=0}^{\infty} \frac{(t-s)^m}{m!}}_{=e^{t-s}}$$

hence

$$\sum_{k=1}^{\infty} e^{-(t-s)} \sum_{l=k}^{\infty} \frac{(t-s)^l}{l!} = t-s \quad \text{and}$$

$$\Phi(t,s) = \begin{bmatrix} 0 & e^{-s}(t-s) \\ 0 & e^{t-s} \end{bmatrix} \checkmark$$

same answer as fund. matrix method.

4. Chen 4.20

$$\dot{x}(t) = \begin{bmatrix} -\sin t & 0 \\ 0 & -\cos t \end{bmatrix} x(t)$$

$$\left.\begin{aligned} \dot{x}_1(t) &= -\sin t \; x_1(t) \;\Rightarrow\; x_1(t) = a e^{\cos t} \\ \dot{x}_2(t) &= -\cos t \; x_2(t) \;\Rightarrow\; x_2(t) = b e^{-\sin t} \end{aligned}\right\}$$ a and b are chosen to satisfy the initial conditions

Fundamental matrix method...

$$\underline{x}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow a = 1, \; b = 0$$

$$\underline{x}_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow a = 0, \; b = 1$$

so the fundamental matrix is

$$\underline{\underline{X}}(t) = \begin{bmatrix} e^{\cos t} & 0 \\ 0 & e^{-\sin t} \end{bmatrix}$$

$$\underline{\underline{X}}^{-1}(s) = \begin{bmatrix} e^{-\cos s} & 0 \\ 0 & e^{+\sin s} \end{bmatrix}$$

STM: $$\boxed{\Phi(t,s) = X(t)X^{-1}(s) = \begin{bmatrix} e^{\cos t - \cos s} & 0 \\ 0 & e^{\sin s - \sin t} \end{bmatrix}}$$