ECE504 HW5 solution

1\. a) $A = \begin{bmatrix} 1 & 4 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

characteristic polynomial : $\det(\lambda I_3 - A) = \begin{vmatrix} \lambda-1 & -4 & 0 \\ 0 & \lambda-2 & 0 \\ 0 & 0 & \lambda-2 \end{vmatrix} = (\lambda-1)(\lambda-2)^2$

algebraic multiplicities : $\lambda_1 = 1,\ r_1 = 1$
$\lambda_2 = 2,\ r_2 = 2$

Now find bases for eigenspaces :

$\underline{E(\lambda_1)}$: $A - \lambda_1 I_3 = \begin{bmatrix} 0 & 4 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ by inspection $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

check : $(A - \lambda_1 I_3)v_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ✓

geometric multiplicty $m_1 = 1$

$\underline{E(\lambda_2)}$: $A - \lambda_2 I_3 = \begin{bmatrix} -1 & 4 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

by inpection $v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 4 \\ 1 \\ 0 \end{bmatrix}$ are both in the nullspace.

geometric multiplicity $m_2 = 2$.

Hence, since $r_i = m_i$ for all i, A is diagonalizable.

If $V = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, then $V^{-1} = \begin{bmatrix} 1 & -4 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$

and $V^{-1}AV = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

continued...

b) $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ characteristic polynomial

$$\det(\lambda I_2 - A) = \begin{vmatrix} \lambda & -1 \\ 0 & \lambda \end{vmatrix} = \lambda^2$$

algebraic multiplicities : $\lambda_1 = 0$, $r_1 = 2$

Now find basis for eiginspace of this e-value:

$\underline{E(\lambda_1)}$: $A - \lambda_1 I_2 = A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

by inspection $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in the nullspace of A.

There can be no more linearly independent vectors in the nullspace of A because $\text{rank}(A) = 1$ and $\text{nullity}(A) + \text{rank}(A) = 2$ here, implying that $\text{nullity}(A) = 1$.

So the geometric multiplicity $m_1 = 1 < r_1$

This matrix is <u>not</u> diagonalizable.

c) $A = I_n$; characteristic polynomial $\det(\lambda I_n - A) = (\lambda - 1)^n$

algebraic multiplicity : $\lambda_1 = 1$, $r_1 = n$

Now find a basis for eigenspace of this evalue:

$E(\lambda_1)$: $A - \lambda_1 I_n$ = $n \times n$ matrix of all zeros.

Hence $v_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, $\ldots$, $v_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$

are all linearly independent and in the nullspace of $A - \lambda_1 I_n$

$\Rightarrow$ geometric multiplicity $m_1 = n = r_1$

$\Rightarrow$ this matrix is diagonalizable (it is already diagonal!)

2. From problem 1, part a, we have

$$V = \begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad V^{-1} = \begin{bmatrix} 1 & -4 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \text{and } V^{-1}AV = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{\Lambda}$$

So $e^{tA} = V^{-1} e^{t\Lambda} V$

$$= \begin{bmatrix} 1 & -4 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^t & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} e^t & 0 & 4e^t \\ 0 & 0 & e^{2t} \\ 0 & e^{2t} & 0 \end{bmatrix} = \boxed{\begin{bmatrix} e^t & 0 & 4e^t - 4e^{2t} \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^{2t} \end{bmatrix} = e^{tA}}$$

similarly $A^{100} = V^{-1} \Lambda^{100} V$

$$= \begin{bmatrix} 1 & -4 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2^{100} & 0 \\ 0 & 0 & 2^{100} \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -4 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 4 \\ 0 & 0 & 2^{100} \\ 0 & 2^{100} & 0 \end{bmatrix}$$

hence $\boxed{A^{100} = \begin{bmatrix} 1 & 0 & 4 - 4\cdot 2^{100} \\ 0 & 2^{100} & 0 \\ 0 & 0 & 2^{100} \end{bmatrix}}$ (check this in Matlab)

For $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ (part b) we can just use the definition of $e^{tA} = \sum_{k=0}^{\infty} \frac{t^k}{k!} A^k = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & t \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \cdots$

hence $\boxed{e^{tA} = \begin{bmatrix} 1 & t \\ 0 & 0 \end{bmatrix}}$

Also $\boxed{A^{100} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}$

3. zero-input response

$$x(t) = \Phi(t,s)\, x(0) = e^{tA} x(0)$$

first experiment: $\begin{bmatrix} 2e^{-2t} - e^{-3t} \\ e^{-3t} \end{bmatrix} = e^{tA} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

second experiment: $\begin{bmatrix} e^{-3t} \\ -e^{-3t} \end{bmatrix} = e^{tA} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

combine:

$$\begin{bmatrix} 2e^{-2t} - e^{-3t} & e^{-3t} \\ e^{-3t} & -e^{-3t} \end{bmatrix} = e^{tA} \underbrace{\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}}_{\text{this matrix is invertible.}}$$

$$\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

So multiply by $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}^{-1}$ on both sides (from the right) to get

$$\frac{1}{2} \begin{bmatrix} 2e^{-2t} - e^{-3t} & e^{-3t} \\ e^{-3t} & -e^{-3t} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = e^{tA}$$

$$\Rightarrow \boxed{e^{tA} = \frac{1}{2} \begin{bmatrix} 2e^{-2t} & 2e^{-2t} - 2e^{-3t} \\ 0 & 2e^{-3t} \end{bmatrix} = \begin{bmatrix} e^{-2t} & e^{-2t} - e^{-3t} \\ 0 & e^{-3t} \end{bmatrix} = \text{STM}}$$

check: $t=0$: $e^{tA} = I_2$ ✓
$\frac{d}{dt} e^{tA} = A e^{tA}$?

remaining problem is to find A such that $e^{tA} =$ (the matrix above)

Since we have e^{-2t} terms and e^{-3t} terms, it should be clear that A must have e-values of $\lambda_1 = -2$ and $\lambda_2 = -3$

Distinct e-values $\Rightarrow$ A is diagonalizable.

continued ...

'Problem 3 continued...

$\therefore$ to find A, we can use the property of STMs

$$\frac{d}{dt}\Phi(t,s) = A(t)\,\Phi(t,s)$$

$$\Rightarrow \frac{d}{dt} e^{tA} = A e^{tA}$$

$$\frac{d}{dt}\begin{bmatrix} e^{-2t} & e^{-2t}-e^{-3t} \\ 0 & e^{-3t} \end{bmatrix} = \begin{bmatrix} -2e^{-2t} & -2e^{-2t}+3e^{-3t} \\ 0 & -3e^{-3t} \end{bmatrix}$$

$$\text{let } A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

$$\text{Then} \quad A e^{tA} = \begin{bmatrix} a_{11} e^{-2t} & a_{11}(e^{-2t}-e^{-3t}) + a_{12} e^{-3t} \\ a_{21} e^{-2t} & a_{21}(e^{-2t}-e^{-3t}) + a_{22} e^{-3t} \end{bmatrix}$$

set these equal to get

$$a_{11} = -2$$
$$a_{21} = 0$$
$$a_{22} = -3$$
$$a_{12} = 1$$

$$\Rightarrow \boxed{A = \begin{bmatrix} -2 & 1 \\ 0 & 3 \end{bmatrix}}$$

4:

4.2 Find unit step response of

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix} x + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u$$

$$y = [\,2 \quad 3\,]\, x$$

Method 1: Laplace transform

$$\hat{g}(s) = [\,2 \quad 3\,] \begin{bmatrix} s & -1 \\ 2 & s+2 \end{bmatrix}^{-1} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= [\,2 \quad 3\,] \frac{1}{s^2+2s+2} \begin{bmatrix} s+2 & 1 \\ -2 & s \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$= \frac{5s}{s^2+2s+2} \, , \qquad \hat{u}(s) = \frac{1}{s}$$

$$\hat{y}(s) = \hat{g}(s)\,\hat{u}(s) = \frac{5s}{(s+1)^2+1} \cdot \frac{1}{s}$$

$$\therefore\ y(t) = 5e^{-t} \sin t$$

Method 2: Using (4.7)

$$\Delta(\lambda) = \det \begin{bmatrix} \lambda & -1 \\ 2 & \lambda+2 \end{bmatrix} = \lambda^2 + 2\lambda + 2$$

$$\lambda = -1 \pm j$$

$$f(\lambda) = e^{\lambda t}, \quad h(\lambda) = \beta_0 + \beta_1 \lambda$$

$$\lambda = -1-j \ : \quad e^{(-1-j)t} = \beta_0 + \beta_1(-1-j)$$

$$\lambda = -1+j \ : \quad e^{(-1+j)t} = \beta_0 + \beta_1(-1+j)$$

$$\Rightarrow \beta_0 = e^{-t} \sin t, \quad \beta_1 = e^{-t}(\sin t + \cos t)$$

$$e^{At} = \beta_0 I + \beta_1 \begin{bmatrix} 0 & 1 \\ -2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} e^{-t}(\sin t + \cos t) & e^{-t} \sin t \\ -2e^{-t} \sin t & e^{-t}(\cos t - \sin t) \end{bmatrix}$$

$u(t) = 1$ for $t \geq 0$.

$$y(t) = [\,2 \quad 3\,] \int_0^t e^{A(t-\tau)} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \cdot 1 \cdot d\tau$$

$$= \int_0^t \left(5e^{-(t-\tau)} \cos(t-\tau) - 5e^{-(t-\tau)} \sin(t-\tau)\right) d\tau$$

Consider

$$5\int_0^t e^{-(t-\tau)} \cos(t-\tau)\, d\tau = -5\int_0^t e^{-(t-\tau)} \frac{d}{d\tau} \sin(t-\tau)\, d\tau$$

$$= -5\left[e^{-(t-\tau)} \sin(t-\tau) \Big|_{\tau=0}^{t} - \int_0^t e^{-(t-\tau)} \sin(t-\tau)\, d\tau \right]$$

Thus we have

$$y(t) = -5\left[e^{-0} \cdot \sin 0 - e^{-t} \sin t \right] = 5e^{-t} \sin t.$$