ECE504 Homework 6 Solution

1. Chen 5.10

$$\dot{x} = \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} x$$

compute e-values...

$$\det(\lambda I - A) = \det \begin{bmatrix} \lambda+1 & 0 & -1 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} = (\lambda+1)\lambda^2$$

$\lambda_1 = -1, \quad r_1 = 1$

$\lambda_2 = 0, \quad r_2 = 2 \quad \rightarrow$ not asymptotically stable

So we need to check the geometric multiplicity of λ_2 to check (marginal) stability...

$$m_2 = \dim(\text{null}(\lambda_2 I - A)) = \dim(\text{null}(A)) = \dim\left(\text{null}\left(\begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}\right)\right) = 2$$

e.g. $v_{21} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ and $v_{22} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$

are linearly independent and both in $\text{null}(\lambda_2 I - A)$.

hence, according to the theorem presented in lecture, this system is (marginally) stable.

2. Chen 5.13

$$x[k+1] = \begin{bmatrix} 0.9 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} x[k]$$

compute e-values : $\lambda_1 = 0.9$, $r_1 = 1$
$\lambda_2 = 1$, $r_2 = 2$

not asymptotically stable

Need to check geometric multiplicity of λ_2 to see if this system is (marginally) stable...

$$m_2 = \dim(\text{null}(\lambda_2 I - A)) = \dim\left(\text{null}\left(\begin{bmatrix} 0.1 & 0 & -1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}\right)\right) = 1$$

only one linearly independent vector in the nullspace.

hence $m_2 = 1 < r_2 = 2 \Rightarrow$ not marginally stable

3. Chen 5.18

This is an if and only if theorem, so we will first assume the equation

$$A^T M + MA + 2\mu M = -N \qquad (1)$$

has a unique positive definite solution for M, given any positive definite matrix N.

Rewrite (1) ...

$$A^T M + MA + \mu I M + \mu M I = -N$$

$$(A^T + \mu I)M + M(A + \mu I) = -N \qquad (2)$$

The Lyapunov theorem states that if N is positive definite and there is a unique positive definite solution for M in (2), then the e-values of $A + \mu I$ must all have real parts less than zero. continued...

Chen 5.18 continued...

But how do the e-values of $A+\mu I$ relate to the e-values of A?

If v_i is an e-vector associated with e-value λ_i then we know that

$$A v_i = \lambda_i v_i$$

This implies that

$$(A+\mu I) v_i = A v_i + \mu v_i = (\lambda_i + \mu) v_i$$

hence v_i is also an e-vector of $A+\mu I$ and has the associated e-value $\gamma_i = \lambda_i + \mu$.

Our previous result implied that $\mathrm{Re}\{\gamma_i\} < 0$ for all i

hence $\mathrm{Re}\{\lambda_i + \mu\} < 0$ for all i

hence $\mathrm{Re}\{\lambda_i\} < -\mu$ for all i

which is the result we wanted (all e-values of A have real part less than $-\mu$).

Now, if we assume all of the e-values of A have real part less than $-\mu$, then all of the e-values of $A+\mu I$ must have real part less than zero.

The Lyapunov stability theorem is an if and only if theorem, so if $A+\mu I$ has all evalues with real part less than zero, then

$$(A+\mu I)^T M + M(A+\mu I) = -N$$

has a unique pos. defn. solution for M given any choice of positive defn. But this can be rewritten as

$$A^T M + MA + 2\mu M = -N,$$

which is the result we wanted.

4. Chen 5.23

You can use the fundamental matrix method to get

$$\Phi(t,s) = \begin{bmatrix} e^{-(t-s)} & 0 \\ 0.2\left(e^{-5t+s} - e^{-4s}\right) & 1 \end{bmatrix}$$

given any $s \geq 0$, the elements of $\Phi(t,s)$ are all bounded for all $t \geq s$. So this system is marginally stable.

To check asymptotic stability, let $t \to \infty$ (fix s)

$$\lim_{t\to\infty} \Phi(t,s) = \begin{bmatrix} 0 & 0 \\ -0.2e^{-4s} & 1 \end{bmatrix}$$

Note that some of the terms in the STM do not go to zero, hence this system is not asymptotically stable.

5.

(5)

(a) For diagonal A we have

$$A = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} = \Lambda$$

so the Lyapunov equation is

$$\Lambda P + P\Lambda = -Q$$

thus

$$\lambda_i p_{ij} + p_{ij}\lambda_j = -q_{ij}$$

and

$$p_{ij} = -\frac{q_{ij}}{\lambda_i + \lambda_j}$$

(b) With A diagonalizable

$$A = T\Lambda T^{-1}$$

where

$$T = \begin{pmatrix} q_1 & \cdots & q_n \end{pmatrix}$$

and q_i is the eigenvector of A corresponding tot λ_i. Thus the Lyapunov equation can be written as

$$\left(T\Lambda T^{-1}\right)^T P + P\left(T\Lambda T^{-1}\right) = -Q$$

or

$$\Lambda\left(T^T P T\right) + \left(T^T P T\right)\Lambda = -T^T Q T$$

Let $\widetilde{P} = T^T P T$ and $\widetilde{Q} = T^T Q T$ then

$$\Lambda\widetilde{P} + \widetilde{P}\Lambda = -\widetilde{Q}$$

and

$$\widetilde{p}_{ij} = -\frac{\widetilde{q}_{ij}}{\lambda_i + \lambda_j}$$

$$P = \left(T^{-1}\right)^T \widetilde{P} T^{-1} \qquad \blacksquare$$