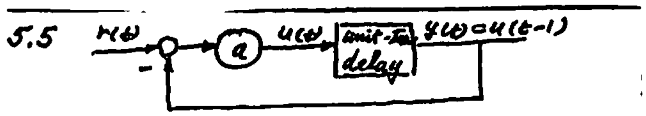
ECE504 Homework #7 Solution

1. Chen 5.5



If $r(t) = \delta(t)$, then

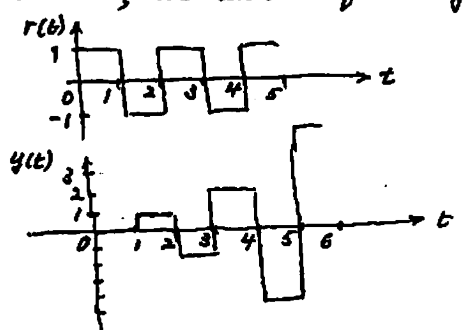
$$g_f(t) = y(t) = a\,\delta(t-1) - a^2\delta(t-2) + a^3\delta(t-3) - a^4\delta(t-4) + \cdots$$

$$\int_0^\infty |g_f(t)|\,dt = |a| + |a|^2 + |a|^3 + \cdots$$

$$= |a|\sum_{i=0}^{\infty} |a|^i = \begin{cases} \dfrac{|a|}{1-|a|} & \text{if } |a|<1 \\ \infty & \text{if } |a| \geq 1 \end{cases}$$

Thus the feedback system is BIBO stable if and only if $|a| < 1$.

For $a=1$, we have the following pair



The bounded input excites an unbounded output.

2. $A = \begin{bmatrix} 0 & \alpha \\ 2 & -1 \end{bmatrix}$ e-values $= \dfrac{-1 \pm \sqrt{1+8\alpha}}{2}$

characteristic poly: $(\lambda - 0)(\lambda+1) - (-2)(-\alpha)$

$= \lambda^2 + \lambda - 2\alpha$

Third criterion for BIBO stability: if A is Hurwitz then the system is BIBO stable.

Hurwitz $\iff Re(\lambda_1) < 0$ and $Re(\lambda_2) < 0$

$\iff Re\left(\sqrt{1+8\alpha}\right) < 1$

$\iff \alpha < 0$.

$\boxed{\text{Hence BIBO stable iff } \alpha < 0}$

Transfer function can also be computed as $\dfrac{\alpha}{s^2+s-2\alpha}$ (minimal realization)

3. This SS system is LTI and can be converted to a TF...

$$\hat{g}(s) = \frac{(s+1)(s-0.5)}{(s+1)^2(s-0.5)} \qquad \text{cancellations...}$$

$$= \frac{1}{s+1} \qquad \text{poles all have negative real part, hence } \underline{\text{BIBO stable}}.$$

This is an example of a system that has an A matrix which is not Hurwitz, yet the system is BIBO stable because $\{A, B, C, D\}$ is not a minimal realization of this TF.

$\boxed{\text{BIBO stable}}$, but not internally stable.

4.

$$Q_r = [B \;\; AB] = \begin{bmatrix} b_1 & b_1 - b_2 \\ b_2 & b_2 - b_1 \end{bmatrix}$$

The set of reachable states is all of $\mathbb{R}^2$ if $\det(Q_r) \neq 0$

$$\det(Q_r) = b_1(b_2 - b_1) - b_2(b_1 - b_2) = b_2^2 - b_1^2$$

So this is a reachable system unless $b_2 = \pm b_1$

- When $b_2 = b_1 \neq 0$, $Q_r = \begin{bmatrix} b_1 & 0 \\ b_1 & 0 \end{bmatrix}$ and a basis for the set of reachable states is then $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$
- When $b_2 = -b_1 \neq 0$, $Q_r = \begin{bmatrix} b_1 & 2b_1 \\ -b_1 & -2b_1 \end{bmatrix}$ and a basis for the set of reachable states is $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$
- When $b_1 = b_2 = 0$, $Q_r = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and there are no reachable states except $x = 0$.
- Otherwise, a basis for the set of reachable states is $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

continued...

problem 4 continued...

Recall that $\{\text{reachable states}\} \subseteq \{\text{controllable states}\}$

Hence, we only need to consider the first 3 cases.

To find the controllable states, note that

$$x[1] = Ax[0] + Bu[0]$$

$$x[2] = Ax[1] + Bu[1] = A^2 x[0] + ABu[0] + Bu[1] = 0$$

$$A^2 = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = 2A$$

So we have $2Ax[0] + ABu[0] + Bu[1] = 0 \leftarrow$ if we can find $u[0]$ and $u[1]$ so that this is true for an $x[0]$, then $x[0]$ is controllable.

$$\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} x_1[0] \\ x_2[0] \end{bmatrix} = \begin{bmatrix} b_1 - b_2 & b_1 \\ b_2 - b_1 & b_2 \end{bmatrix}\begin{bmatrix} u[0] \\ u[1] \end{bmatrix}$$

<u>when $b_2 = b_1$ and $b_1 \neq 0$</u> then
$$b_1 u[1] = 2x_1[0] - 2x_2[0]$$
$$b_1 u[1] = -2x_1[0] + 2x_2[0]$$

hence the state $x[0]$ is controllable only if $2x_1[0] - 2x_2[0] = -2x_1[0] + 2x_2[0]$
$\iff x_1[0] = x_2[0]$ (same as reachable states)

<u>when $b_2 = -b_1$ and $b_1 \neq 0$</u> then

$$2b_1 u[0] + b_1 u[1] = 2x_1[0] - 2x_2[0]$$

$$-2b_1 u[0] - b_1 u[1] = -2x_1[0] + 2x_2[0]$$

hence the state $x[0]$ is controllable only if

$$2x_1[0] - 2x_2[0] = -(-2x_1[0] + 2x_2[0])$$

which is true for any $x_1[0]$ and $x_2[0]$.

Hence, when $b_1 = -b_2$, the set of controllable states $= \mathbb{R}^2$
(this is different than the set of reachable states).

<u>when $b_2 = b_1 = 0$</u> then

$$0 = 2x_1[0] - 2x_2[0]$$

$$0 = -2x_1[0] + 2x_2[0]$$

This is satisfied if $x_1[0] = x_2[0]$. Hence, a basis for the set of controllable states is $\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$ when $b_1 = b_2 = 0$.

5. $Q_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} c_1 & c_2 \\ c_1 - c_2 & c_2 - c_1 \end{bmatrix}$

unobservable states $\rightarrow$ null (Q_o)

Do Gaussian elimination: (assume, for now, $c_1 \neq 0$)

$$\left[\begin{array}{cc|c} c_1 & c_2 & 0 \\ c_1 - c_2 & c_2 - c_1 & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} c_1 & c_2 & 0 \\ 0 & \frac{c_2(c_2-c_1)}{c_1} + \frac{c_1(c_2-c_1)}{c_1} & 0 \end{array}\right]$$

$$\rightarrow \left[\begin{array}{cc|c} c_1 & c_2 & 0 \\ 0 & \frac{c_2^2 - c_1^2}{c_1} & 0 \end{array}\right] \rightarrow \left[\begin{array}{cc|c} 1 & \frac{c_2}{c_1} & 0 \\ 0 & \left(\frac{c_2}{c_1}\right)^2 - 1 & 0 \end{array}\right]$$

so $\dim(\text{null}(Q_o)) > 0 \Leftrightarrow \left(\frac{c_2}{c_1}\right)^2 = 1 \Leftrightarrow c_2 = \pm c_1$

- case 1 : $c_2 = c_1 \neq 0$

  $Q_o = \begin{bmatrix} c_1 & c_1 \\ 0 & 0 \end{bmatrix}$ a basis for null (Q_o) is then $\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$

- case 2: $c_2 = -c_1 \neq 0$

  $Q_o = \begin{bmatrix} c_1 & -c_1 \\ 2c_1 & -2c_1 \end{bmatrix}$ a basis for null (Q_o) is then $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

- case 3 : $c_1 = 0$

  $Q_o = \begin{bmatrix} 0 & c_2 \\ -c_2 & c_2 \end{bmatrix}$ full rank unless $c_2 = 0$

  $c_1 = c_2 = 0 \Rightarrow Q_o = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ a basis for null (Q_o) is then $\left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$

- Case 4: otherwise

  Q_o is full rank and the only element in null (Q_o) is the zero vector. In this case, and only this case, the system is observable.