# ECE504 Fall 2009 HW8 Solution

1. Chen 6.1 This is a continuous time system, so the set of controllable states will be the same as the set of reachable states.

$$Q_r = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 3 \end{bmatrix}$$ range$(Q_r) = \mathbb{R}^3$ $\Rightarrow$ "controllable"

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ -1 & -2 & -1 \\ 1 & 2 & 1 \end{bmatrix}$$ rank$(Q_o) = 1$ $\Rightarrow$ can find two linearly indep vectors in null$(Q_o)$

$$v_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} 0 \\ -1 \\ 2 \end{bmatrix}$$ are both in null$(Q_o)$

so $\alpha v_1 + \beta v_2$ is also in null$(Q_o)$

since null$(Q_o)$ is more than just the zero vector, this system is not observable.

2. Chen 6.3

Suppose $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Then $\begin{bmatrix} B & AB \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{rank}=1}$ and $\begin{bmatrix} AB & A^2B \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}_{\text{rank}=0}$

So it is not true that the rank of $\begin{bmatrix} B & AB & \cdots & A^{n-1}B \end{bmatrix} = Q_1$ is equal to the rank of $\begin{bmatrix} AB & A^2B & \cdots & A^{n-1}B \end{bmatrix} = Q_2$

Note that $Q_2 = A\,Q_1$ for $A \in \mathbb{R}^{n \times n}$

The rank of $Q_2$ will only be the same as the rank of $Q_1$ if $A$ is a one-to-one mapping, i.e. $A$ is nonsingular.

3. Chen 6.8

$$\dot{x}(t) = \begin{bmatrix} -1 & 4 \\ 4 & -1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} x(t)$$

compute TF : $\hat{g}(s) = C(sI - A)^{-1}B + D = \dfrac{2s+10}{s^2+2s-15}$

$$= \frac{2\cancel{(s+5)}}{\cancel{(s+5)}(s-3)}$$

$\hat{g}(s) = \dfrac{2}{s-3}$ , no further pole-zero cancellations possible.

Now realize this system using methods covered in week 2 of course...

$\hat{g}(s) = \dfrac{\hat{N}(s)}{\hat{D}(s)} = \dfrac{\hat{y}(s)}{\hat{u}(s)}$ $\quad \hat{N}(s) = 2$, $\hat{D}(s) = s-3$

$$\hat{v}(s) = \frac{1}{\hat{D}(s)} \hat{u}(s) \Rightarrow (s-3)\hat{v}(s) = \hat{u}(s)$$

$$\dot{v}(t) - 3v(t) = u(t)$$

$$\boxed{\dot{v}(t) = 3v(t) + u(t)}$$

$$\hat{y}(s) = \hat{N}(s)\hat{v}(s) \Rightarrow \boxed{y(t) = 2v(t)}$$

So, letting $x(t) = v(t)$, we have

$$\dot{x}(t) = Ax(t) + Bu(t)$$
$$y(t) = Cx(t) + Du(t)$$

with $A = 3$, $B = 1$, $C = 2$, $D = 0$.

This system is <u>minimal</u>, hence it must be <u>controllable</u> and observable.

Check : $Q_r = [B] = 1$ $\quad$ rank = 1 $\Rightarrow$ controllable

$Q_o = [C] = 2$ $\quad$ rank = 1 $\Rightarrow$ observable.

4. Chen 6.15

Note that the $A$ matrix in this problem is block diagonal

$$A^k = \begin{bmatrix} A_1^k & & \\ & A_2^k & \\ & & A_3^k \end{bmatrix} \qquad A_1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad A_2 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad A_3 = [1]$$

Hence $A^k = \begin{bmatrix} 1 & k & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & k & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

$$Q_r = \begin{bmatrix} B & AB & A^2B & A^3B & A^4B \end{bmatrix} \in \mathbb{R}^{5 \times 10}$$

$$= \left[\begin{array}{cc:cc:cc:cc:cc} b_{11} & b_{12} & b_{11}+b_{21} & b_{12}+b_{22} & b_{11}+2b_{21} & b_{12}+2b_{22} & b_{11}+3b_{21} & b_{12}+3b_{22} & b_{11}+4b_{21} & b_{12}+4b_{22} \\ b_{21} & b_{22} & b_{21} & b_{22} & b_{21} & b_{22} & b_{21} & b_{22} & b_{21} & b_{22} \\ b_{31} & b_{32} & b_{31}+b_{41} & b_{32}+b_{42} & b_{31}+2b_{41} & b_{32}+2b_{42} & b_{31}+3b_{41} & b_{32}+3b_{42} & b_{31}+4b_{41} & b_{32}+4b_{42} \\ b_{41} & b_{42} & b_{41} & b_{42} & b_{41} & b_{42} & b_{41} & b_{42} & b_{41} & b_{42} \\ b_{51} & b_{52} & b_{51} & b_{52} & b_{51} & b_{52} & b_{51} & b_{52} & b_{51} & b_{52} \end{array}\right]$$

In order for $Q_r$ to have rank $= 5$, all rows must be linearly independent. Look at the 2nd, 4th, and 5th rows...
These are linearly independent only if

$[b_{21} \;\; b_{22}]$, $[b_{41} \;\; b_{42}]$, and $[b_{51} \;\; b_{52}]$ are linearly indep.

These can't be linearly indep, hence $\text{rank}(Q_r) < 5$

$\Rightarrow$ not controllable for any choice of $B$.

As for observability, the same procedure can be followed to compute $Q_o \in \mathbb{R}^{15 \times 5}$

To get $\text{rank}(Q_o) = 5$, we need $\begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \end{bmatrix}$, $\begin{bmatrix} c_{13} \\ c_{23} \\ c_{33} \end{bmatrix}$, and $\begin{bmatrix} c_{15} \\ c_{25} \\ c_{35} \end{bmatrix}$

to be linearly independent. This is easy: $c_{11} = 1$, $c_{23} = 1$, $c_{35} = 1$ and all other terms $= 0$.

Hence $C = \begin{bmatrix} 1 & a & 0 & d & 0 \\ 0 & b & 1 & e & 0 \\ 0 & c & 0 & f & 1 \end{bmatrix}$ causes the system to be observable for any $a, b, c, d, e, f$.

5. Chen 7.2

$$\hat{g}(s) = \frac{s-1}{(s^2-1)(s+2)} = \frac{s-1}{s^3+2s^2-s-2} = \frac{\beta_1 s^2+\beta_2 s+\beta_3}{s^3+\alpha_1 s^2+\alpha_2 s+\alpha_3}$$

Use observable canonical form... (page 188)

$\alpha_1 = 2$, $\alpha_2 = -1$, $\alpha_3 = -2$ ; $\beta_1 = 0$, $\beta_2 = 1$, $\beta_3 = -1$

$$\dot{x}(t) = \begin{bmatrix} -2 & 1 & 0 \\ 1 & 0 & 1 \\ 2 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} x(t) + 0u(t)$$

$$Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 5 & -2 & 1 \end{bmatrix}, \text{ rank}=3 \Rightarrow \text{observable}$$

$$Q_r = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 0 & 1 & -3 \\ 1 & -1 & 1 \\ -1 & 0 & 2 \end{bmatrix}$$

note that column 3 is equal to $-2 \times$ column 1 $-3 \times$ colum 2

rank = 2 $\Rightarrow$ <u>not controllable</u>.

This is not surprising since $\hat{g}(s)$ had a pole-zero cancellation and the resulting SS description was not minimal. Not minimal implies not controllable or not observable, or both.