# ECE504 Homework 9 Solution

1. $\hat{g}(s) = \dfrac{s}{s^2+1}$

a) minimal realization (use our old trick)

$$(s^2+1)\,\hat{v}(s) = \hat{u}(s)$$

$$\ddot{v}(t) + v(t) = u(t)$$

let $x(t) = \begin{bmatrix} \dot{v}(t) \\ v(t) \end{bmatrix}$

Then $$\boxed{\dot{x}(t) = \begin{bmatrix} \ddot{v}(t) \\ \dot{v}(t) \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t)}$$

$\hat{y}(s) = s\,\hat{v}(s) \Rightarrow y(t) = \dot{v}(t)$

Then $$\boxed{y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) + [0]\, u(t)}$$

$$\left.\begin{array}{l} Q_r = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \text{reachable} \\ Q_o = \begin{bmatrix} C \\ CA \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \text{observable} \end{array}\right\} \Rightarrow \text{minimal.}$$

b) realization that is observable but not reachable/controllable.

Procedure: . . .
1. Make new TF with a pole/zero cancellation.
2. Realize the TF in observable canonical form

$$\hat{g}(s) = \frac{s^2}{s^3+s}$$

see chen p. 188 for observable canonical form...

$$A = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad D = 0 \quad \left(\text{check TF!}\right)$$

check observability: $Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \Rightarrow$ rank=3 observable

check reachability: $Q_r = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ rank=2 not reachable

c) same idea for a system that is reachable but not observable...

$$\hat{g}(s) = \frac{s^2}{s^3+s}$$

controllable canonical form:

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad D = 0$$

(check TF!)

check observability: $Q_o = \begin{bmatrix} C \\ CA \\ CA^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}$ rank = 2 $\Rightarrow$ not observable

check reachability: $Q_r = \begin{bmatrix} B & AB & A^2B \end{bmatrix} = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ rank = 3 $\Rightarrow$ reachable.

2. $$\hat{G}(s) = \begin{bmatrix} \frac{s+1}{s+2} & \frac{1}{s+3} \\ \frac{s}{s+1} & \frac{s+1}{s+2} \end{bmatrix}$$

minors of order 1: $\left\{ \frac{s+1}{s+2}, \frac{1}{s+3}, \frac{s}{s+1}, \frac{s+1}{s+2} \right\}$

minors of order 2: $\left\{ \frac{(s+1)^2}{(s+2)^2} - \frac{s}{(s+1)(s+3)} \right\}$

$$= \left\{ \frac{(s+1)^2 - s(s+2)^2}{(s+1)(s+2)^2(s+3)} \right\}$$

least common denominator is $(s+1)(s+2)^2(s+3)$

Hence the Mc Millan degree of this system is 4.

Easiest way to construct a minimal realization is to construct 4 mini-systems (each uncoupled from the other).

i) $\frac{s+1}{s+2} = \frac{-1}{s+2} + 1 \Rightarrow A = -2 \quad B = 1 \quad C = -1 \quad D = 1$

ii) $\frac{1}{s+3} \Rightarrow A = -3,\ B = 1,\ C = 1,\ D = 0$

iii) $\frac{s}{s+1} = \frac{-1}{s+1} + 1 \Rightarrow A = -1,\ B = 1,\ C = -1,\ D = 1$

iv) $\frac{s+1}{s+2} \Rightarrow A = -2,\ B = 1,\ C = -1,\ D = 1$ (as shown above)

continued...

Hence

$$\dot{x}(t) = \begin{bmatrix} -2 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} u(t)$$

$$y(t) = \begin{bmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix} x(t) + \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} u(t)$$

- can confirm that this realizes all of the TFs using Matlab ss2tf
- can confirm that this is a minimal realization by using Matlab function minreal (or checking $Q_o$ and $Q_r$)

3. Chen 8.6

To make $\hat{g}(s) = \dfrac{(s-1)(s+2)}{(s+1)(s-2)(s+3)} = \dfrac{s^2+s-2}{s^3+2s^2-5s-6}$

become $\hat{g}_f(s) = \dfrac{1}{s+3}$ after feedback, we need to move the e-values of the system to $+1, -2$, and $-3$.

Suppose the system was realized in controllable canonical form:

$$A = \begin{bmatrix} -2 & 5 & 6 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad C = \begin{bmatrix} 1 & 1 & -2 \end{bmatrix}$$

We know that this system is reachable/controllable, so we can place the evalues anywhere we want with state feedback.

you can use any of the techniques developed in lecture to find $F = [2 \quad 6 \quad 0]$

check: $\text{eig}(A - BF) = \{-3, -2, +1\}$ ✓

The resulting TF is $\hat{g}(s) = \dfrac{1}{s+3}$ as desired

The resulting system with feedback is BIBO stable but not asymptotically stable because of the e-value $= 1$.

4. Chen 8.8. System is reachable.

You can use any of the methods covered in lecture to find $F = [1 \quad 5 \quad 2]$.

The system with state feedback is then

$$x[k+1] = (A - BF)\,x[k] + Bu[k]$$

$$x[k+1] = \begin{bmatrix} 0 & -4 & -4 \\ 0 & 1 & 1 \\ -1 & -5 & -1 \end{bmatrix} x[k] + \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u[k]$$

$$y[k] = [2 \quad 0 \quad 0]\,x[k] + 0\,u[k]$$

check: e-values of $A - BF$ are all zero.

zero input response: $x[k] = (A-BF)^k\, x[0]$

Note that $(A - BF)^3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Hence $x[k] = 0$ for all $k \geq 3$, for any initial state, which is to be expected if all of the e-values are equal to zero.