ECE504 Midterm Exam Solution

Causal, lumped,

1. Linear time invariant systems can be represented both with transfer functions and state-space descriptions.

State space only:
- time-varying systems
- non-linear systems
- systems with non-zero initial conditions (not relaxed)

transfer function only:
- distributed systems
- non-causal systems, e.g. $y[k] = u[k+1]$
  $\hat{g}(z) = z$ → no SS description possible

2. a) $u(t) - Ry(t) - x(t) = 0$ (1)

$y(t) = C\,\dfrac{dx(t)}{dt}$ (2)

combine (1) and (2) to get

$u(t) - RC\dfrac{dx(t)}{dt} - x(t) = 0$ (3)

rearrange

$$\frac{dx(t)}{dt} = \underbrace{\frac{-1}{RC}}_{A} x(t) + \underbrace{\frac{1}{RC}}_{B} u(t) \qquad (4)$$

← This is the state update equation

Equation (1) is the output equation (rearrange)

$$y(t) = \underbrace{-\frac{1}{R}}_{C} x(t) + \underbrace{\frac{1}{R}}_{D} u(t) \qquad (5)$$

b) This is just a scalar system, so the STM is just

$$\Phi(t,s) = \exp\left\{ \int_s^t -\frac{1}{RC}\, d\tau \right\} = \exp\left\{ \frac{-(t-s)}{RC} \right\}$$

Hence, the response of this system to a unit-step input is

$$x(t) = \Phi(t,0)\, x(0) + \int_0^t \Phi(t,\tau) \frac{1}{RC} \cdot 1 \, d\tau$$

$$= 2e^{-t/RC} + \frac{1}{RC} e^{-t/RC} \underbrace{\int_0^t e^{\tau/RC}\, d\tau}_{RC\left(e^{t/RC} - 1\right)}$$

so

$$x(t) = 2e^{-t/RC} + \left(1 - e^{-t/RC}\right) = e^{-t/RC} + 1.$$

hence

$$y(t) = -\frac{1}{R} x(t) + \frac{1}{R} u(t)$$

$$= -\frac{1}{R} e^{-t/RC} - \frac{1}{R} + \frac{1}{R} \qquad t \geq 0$$

$$\boxed{y(t) = -\frac{1}{R} e^{-t/RC}} \qquad t \geq 0$$

3. $$A(t) = \begin{bmatrix} -1 & e^{-2t} \\ 0 & 1 \end{bmatrix}$$

$$\dot{x}_2(t) = x_2(t) \Rightarrow x_2(t) = c e^{t}$$

$$\dot{x}_1(t) = -x_1(t) + e^{-2t} x_2(t) = -x_1(t) + c e^{-t}$$

try $x_1(t) = e^{-t}(ct + d)$

$$\dot{x}_1(t) = -e^{-t}(ct+d) + ce^{-t} \quad \checkmark$$

so we have $$\underline{x}(t) = \begin{bmatrix} e^{-t}(ct+d) \\ c e^{t} \end{bmatrix}$$

first initial condition: $\begin{bmatrix} 1 \\ 0 \end{bmatrix} \Rightarrow \begin{matrix} d = 1 \\ c = 0 \end{matrix}$ check: $\begin{bmatrix} e^{-0}(0+1) \\ 0e^{0} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

second initial condition $\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow \begin{matrix} d = 0 \\ c = 1 \end{matrix}$ check: $\begin{bmatrix} e^{-0}(0+0) \\ 1e^{0} \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Hence $\underline{\underline{X}}(t) = \begin{bmatrix} e^{-t} & te^{-t} \\ 0 & e^{t} \end{bmatrix}$ and $\underline{\underline{X}}^{-1}(s) = \begin{bmatrix} e^{s} & -se^{-s} \\ 0 & e^{-s} \end{bmatrix}$

and $$\boxed{\Phi(t,s) = \underline{\underline{X}}(t)\underline{\underline{X}}^{-1}(s) = \begin{bmatrix} e^{-(t-s)} & (t-s)e^{-(t+s)} \\ 0 & e^{t-s} \end{bmatrix}}$$

Check: $\Phi(t,t) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \checkmark$

$$\frac{d}{dt}\Phi(t,s) = \begin{bmatrix} -e^{-(t-s)} & e^{-(t+s)} - (t-s)e^{-(t+s)} \\ 0 & e^{t-s} \end{bmatrix}$$

$$A(t)\Phi(t,s) = \begin{bmatrix} -1 & e^{-2t} \\ 0 & 1 \end{bmatrix}\begin{bmatrix} e^{-(t-s)} & (t-s)e^{-(t+s)} \\ 0 & e^{t-s} \end{bmatrix} = \begin{bmatrix} -e^{-(t-s)} & -(t-s)e^{-(t+s)} + e^{-(t+s)} \\ 0 & e^{t-s} \end{bmatrix}$$

$\checkmark$

4. a) Recall that A is invertible if and only if $\det(A) \neq 0$.

$\det(A) = \cos^2\theta + \sin^2\theta = 1 \neq 0$ for all θ.

Hence A is always invertible.

b) A is diagonalizable if and only if the geometric multiplicity of every e-value is the same as its algebraic multiplicity.

e-values:

$$\det(\lambda I - A) = (\lambda - \cos\theta)^2 + \sin^2\theta = 0$$

$$\lambda^2 - 2\lambda\cos\theta + \cos^2\theta + \sin^2\theta = 0$$

$$\lambda^2 - 2\lambda\cos\theta + 1 = 0$$

roots are $\dfrac{2\cos\theta \pm \sqrt{4\cos^2\theta - 4}}{2} = \cos\theta \pm \sqrt{\cos^2\theta - 1}$

Note that the e-values are always distinct except when $\theta = 0$ or $\theta = \pi$. We need to look at these two cases separately

i) $\theta = 0 \Rightarrow A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\lambda_1 = 1$, $r_1 = 2$, $s = 1$

geometric multiplicity: $(\lambda_1 I - A) = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$m_1 = 2$. so diagonalizable

ii) $\theta = \pi$ $A = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ $\lambda_1 = -1$, $r_1 = 2$, $s = 1$

geometric multiplicity $= 2$ here as well. so diagonalizable.

iii) $\theta \neq 0$ and $\theta \neq \pi \Rightarrow$ distinct e-values $\Rightarrow$ diagonalizable

so A is always diagonalizable