ECE531 Lecture 10b: Dynamic Parameter Estimation: System Model

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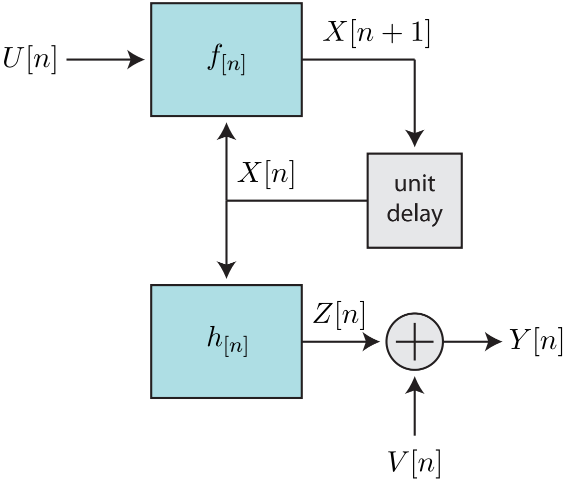
Introduction

- So far, we have only considered estimation problems with fixed parameters, e.g.
  - $Y_k \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\theta, \sigma^2)$.
  - $Y_k = a\cos(\omega k + \phi) + W_k$ for $\theta = [a, \phi, \omega]^\top$.
  - $Y_k \overset{\text{i.i.d.}}{\sim} \theta e^{-\theta y_k}$.
- Many problems require us to estimate **dynamic** or time-varying parameters, e.g.
  - Radar (position and velocity of target changing over time)
  - Communications (amplitude and phase of signal changing over time)
  - Stock market (price of shares next week changing over time)
- Step 1: We need to develop a general model for
  - How time-varying parameters evolve over time and
  - How observations are generated according from the parameter state.
- Step 2: We need to develop good techniques for estimating dynamic parameters. We can expect these techniques to be based on our good static parameter estimators, i.e. MVU and/or ML, and also use some knowledge of the time-varying model.

Discrete Time Model for Dynamic Parameters



The time index is denoted as $n = 0, 1, \ldots$. All vectors are considered to be random unless otherwise specified.

$$
\begin{aligned}
U[n] &\in \mathbb{R}^s \\
X[n] &\in \mathbb{R}^m \\
Z[n] &\in \mathbb{R}^k \\
V[n] &\in \mathbb{R}^k \\
Y[n] &\in \mathbb{R}^k \\
f_{[n]} &: \mathbb{R}^s \times \mathbb{R}^m \mapsto \mathbb{R}^m \\
h_{[n]} &: \mathbb{R}^m \mapsto \mathbb{R}^k
\end{aligned}
$$

Notation and Terminology

- $U[n]$ is the dynamical system "input", $n = 0, 1, \dots$.
- $X[n]$ is the "state" of the dynamical system, $n = 0, 1, \dots$.
- $Z[n]$ is the dynamical system "output", $n = 0, 1, \dots$.
- $f_{[n]}$ is a time varying function that updates the state based on the current state and the current input, i.e.

$$X[n+1] = f_{[n]}(X[n], U[n])$$

- $h_{[n]}$ is a time varying function that generates the current output based on the current state, i.e.

$$Z[n] = h_{[n]}(X[n])$$

- $V[n]$ is the "measurement noise" $n = 0, 1, \dots$.
- $Y[n]$ is the "observation" $n = 0, 1, \dots$.

Example: One-Dimensional Motion

Suppose we have a particle moving on a line with position x and velocity v updated according to

$$\begin{aligned} x[n+1] &= x[n] + Tv[n] \\ v[n+1] &= v[n] + Ta[n] \end{aligned}$$

where $a[n]$ represents the acceleration and T is the time between samples. The implicit assumption here is that T is small enough such that the discrete time model holds.

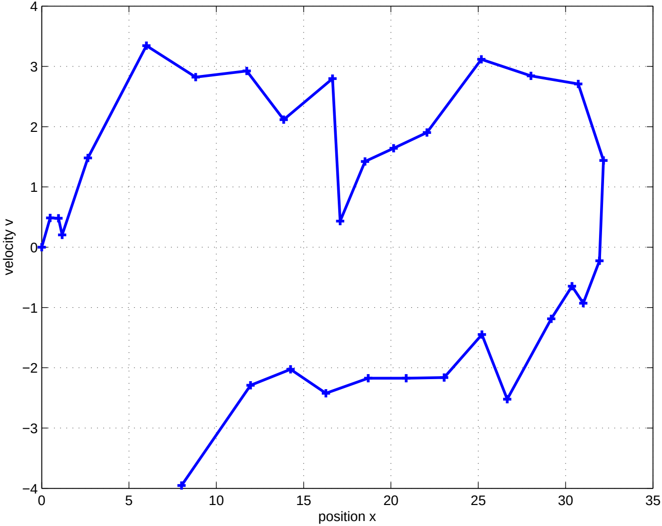
- The state $X = [x, v]^\top$.
- The input $U = a$.
- The state update equation can be written as

$$X[n+1] = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix} X[n] + \begin{bmatrix} 0 \\ T \end{bmatrix} U[n]$$

- Suppose we observe the position of the particle in noise. Then

$$Y[n] = \begin{bmatrix} 1 & 0 \end{bmatrix} X[n] + V[n]$$

Example: One-Dimensional Motion (White Gaussian Input)



Remarks

- In these types of dynamical systems, $X[k]$ is completely determined by the earlier state $X[\ell]$, $l < k$, and the inputs $\{U[\ell], \dots, U[k-1]\}$.
- The state at time ℓ completely summarizes the system in the sense that you don't need to know the details of what happened prior to time ℓ if you know $X[\ell]$.
- We are going to study problems in which we wish to estimate the dynamic state $X[\ell]$ given a sequence of observations $Y[0], \dots, Y[k]$. These problems can be categorized into three types:
  1. **Filtering**: $\ell = k$ (estimate the current state)
  2. **Prediction**: $\ell > k$ (estimate a future state)
  3. **Smoothing**: $\ell < k$ (estimate a previous state)
- Note that the notation in dynamic parameter estimation problems is different than static parameter estimation problems: To be consistent with your textbook, $X[\ell]$ represents the quantity we wish to estimate.

Restriction 1: Squared Error Cost Assignment

We will only consider the squared error cost assignment, i.e.

$$\text{MSE} \;=\; \mathrm{E}\left\{\|\hat{X}[\ell] - X[\ell]\|_2^2\right\}$$

where $\hat{X}[\ell]$ is the estimate of the state $X[\ell]$. We assume that we know the joint distribution of the input $U[n]$ and the distribution of the initial state $X[0]$.

Given the observation $Y[0], \dots, Y[k]$, what estimator $\hat{X}[\ell]$ minimizes the MSE?

Hint: Is this Bayesian estimation or non-random parameter estimation?

Memory and Computational Requirements

- We are often interested in producing state estimates in real time, i.e., upon receiving the observation $Y[k]$, we estimate $X[\ell]$; upon receiving observation $Y[k+1]$, we estimate $X[\ell+1]$; and so on.
- In general, computing

$$\hat{X}_{\text{mmse}}[\ell] \quad = \quad \mathrm{E}\left\{X[\ell] \,|\, Y[0], \dots, Y[k]\right\}$$

  would require us to
  - Keep all the past observations in memory
  - Compute the estimate as a function of all the past observations
- In other words, the general dynamic state estimation problem has linearly growing memory and computational burdens.
- Additional restrictions are going to be necessary to make the dynamic state estimation problem computationally feasible.

Restriction 2: Linear Dynamical Model

We are going to restrict our attention to systems with state update equations and output equations of the form

$$\begin{aligned} X[n+1] &= F[n]X[n] + G[n]U[n] \qquad n = 0, 1, \dots \\ Y[n] &= H[n]X[n] + V[n] \qquad n = 0, 1, \dots \end{aligned}$$

where, for each n, $F[n] \in \mathbb{R}^{m \times m}$, $G[n] \in \mathbb{R}^{m \times s}$, and $H[n] \in \mathbb{R}^{k \times m}$.

- We've already seen that one-dimensional motion fits within this linear model.
- The same is true for two- and three-dimensional motion.
- Many nonlinear systems can approximately fit in this model by linearizing f and h around a nominal state (Taylor series expansion).

Linear Dynamical Model

$$
\begin{aligned}
X[n+1] &= F[n]X[n] + G[n]U[n] \\
&= F[n]\left(F[n-1]X[n-1] + G[n-1]U[n-1]\right) + G[n]U[n] \\
&= \text{etc.}
\end{aligned}
$$

Repeating this process leads to an expression for the state at time $n+1$ in terms of the initital state $X[0]$ and the inputs $U[0], U[1], \dots$:

$$
X[n+1] = \left\{\prod_{k=0}^{n} F[n-k]\right\} X[0] + \sum_{j=0}^{n} \left\{\prod_{k=0}^{n-j-1} F[n-k]\right\} G[j]U[j]
$$

where

$$
\prod_{k=0}^{t} F[n-k] := F[n]F[n-1]F[n-2]\cdots F[n-t]
$$

is called the state transition matrix from time $n-t$ to time $n+1$.

Linear Dynamical Model: Time Invariant Case

When

$$\begin{aligned} F[n] &\equiv F \\ G[n] &\equiv G \end{aligned}$$

then the discrete time dynamical system is time invariant and the state at time $n+1$ is simply

$$X[n+1] = F^{n+1}X[0] + \sum_{j=0}^{n} F^{n-j}G[j]U[j]$$

Restriction 3: Gaussian Input and Measurement Noise

We will revisit this restriction later, but for now we will only consider systems in which the input sequence $U[0], U[1], \dots$ and the measurement noise sequence $V[0], V[1], \dots$ are independent sequences of independent zero mean Gaussian random vectors, i.e.

$$\begin{aligned}
\mathrm{E}\{U[k]\} &= 0 \\
\mathrm{E}\{U[k]U^\top[j]\} &= \begin{cases} Q[k] & k = j \\ 0 & \text{otherwise} \end{cases} \\
\mathrm{E}\{V[k]\} &= 0 \\
\mathrm{E}\{V[k]V^\top[j]\} &= \begin{cases} R[k] & k = j \\ 0 & \text{otherwise} \end{cases} \\
\mathrm{E}\{U[k]V^\top[j]\} &= 0 \qquad \text{for all } j \text{ and } k
\end{aligned}$$

We also assume that the initial state $X[0] \sim \mathcal{N}(m[0], \Sigma[0])$ is a Gaussian random vector independent of $U[0], U[1], \dots$ and $V[0], V[1], \dots$.

Final Remarks and Preview

- As we will see in the development of the Discrete-Time Kalman-Bucy filter, restrictions 2 and 3 will allow us to write the MMSE estimator

$$\hat{X}_{\mathsf{mmse}}[\ell] = \mathrm{E}\left\{X[\ell] \,|\, Y[0], \dots, Y[k]\right\}$$

as

$$\hat{X}_{\mathsf{mmse}}[\ell] = \mathrm{E}\left\{X[\ell] \,|\, \hat{X}[\ell-1], Y[k]\right\}$$

without any loss of optimality.
- In other words, in our restricted linear/Gaussian model, the optimal state estimator only depends on the previous state estimate and the current observation.
- Unlike the general dynamic state estimation problem, the memory requirements and computational burden do not grow in time.
- We will also see that the discrete-time Kalman-Bucy filter possesses certain types of optimality even when restrictions 2 and 3 do not hold.