ECE 531

Midterm exam Solution

Spring 2009

## Problem 1:

a) This is a simple binary HT problem.

b) states: $x_0 =$ "good"
$x_1 =$ "defective"

observations: $y_0 =$ "green"
$y_1 =$ "yellow"
$y_2 =$ "red"

conditional distributions:

$$P_0(y) = \begin{cases} p & y = y_0 \\ 1-p & y = y_1 \\ 0 & y = y_2 \end{cases}$$

$$P_1(y) = \begin{cases} 0 & y = y_0 \\ 1-p & y = y_1 \\ p & y = y_2 \end{cases}$$

hypotheses: $\mathcal{H}_0 : x = x_0$ (good widget)
$\mathcal{H}_1 : x = x_1$ (defective widget)

c) There are 8 deterministic decision rules

$D = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ always decide $\mathcal{H}_0$

$\vdots$

$D = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ always decide $\mathcal{H}_1$

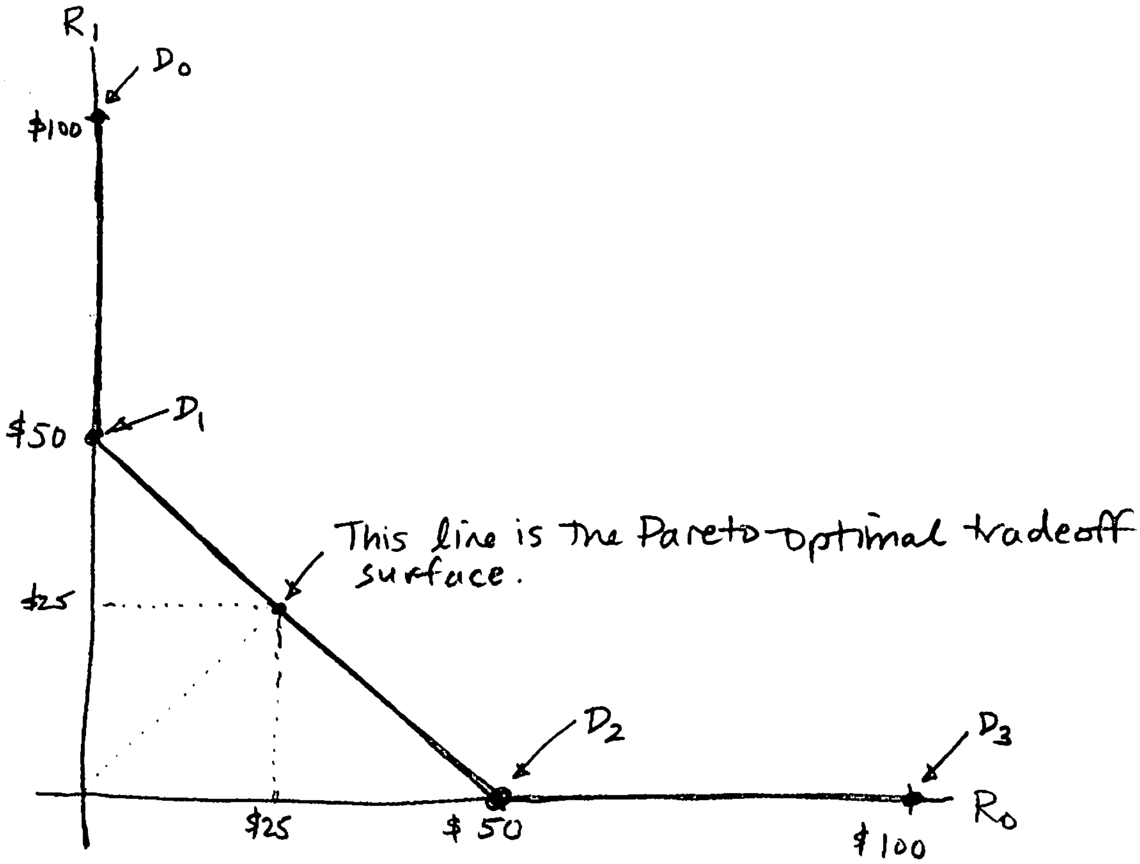
d) We can save some time here by realizing that there are only four good deterministic decision rules:

$D_0 = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$ always decide $\mathcal{H}_0$

$D_1 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ decide $\mathcal{H}_1$ if widget tester shows "red" otherwise decide $\mathcal{H}_0$.

$D_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ decide $\mathcal{H}_0$ if widget tester shows "green" otherwise decide $\mathcal{H}_1$

$D_3 = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ always decide $\mathcal{H}_1$




$R_0(D_1)$ = expected cost of deciding $\mathcal{H}_1$ when true state is $x_0$
$= 0$

$R_1(D_1)$ = expected cost of deciding $\mathcal{H}_0$ when true state is $x_1$
$= (1-p) \cdot 100 = \$50$

Similarly $R_0(D_2) = \$50$ and $R_1(D_2) = 0$

e) From the plot in part (d), we note that a randomized decision rule can achieve the risk trade off $R_0 = R_1 = \$25$.

This randomized decision rule is

$$\boxed{D^* = \frac{1}{2} D_1 + \frac{1}{2} D_2 = \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 1 \end{bmatrix}}$$

check

$$R_0(D^*) = \frac{1}{2}(1-p) \cdot \$100 = \$25$$

$$R_1(D^*) = \frac{1}{2}(1-p) \cdot \$100 = \$25. \quad \checkmark$$

This is a minimax decision rule (equal risks).

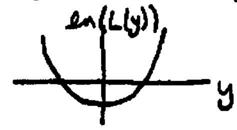
2.

This is a simple binary HT problem. All of the optimal decision rules involve computing the likelihood ratio:

$$L(y) = \frac{p_1(y)}{p_0(y)} = \frac{\frac{1}{2}\exp(-y^2/8)}{\exp(-y^2/2)} \quad \begin{matrix} \leftarrow \sigma = 1 \\ \leftarrow \sigma = 2 \end{matrix}$$

$$= 2\exp\left(\frac{4y^2}{8} - y^2/8\right) = \frac{1}{2}\exp\left(\frac{3y^2}{8}\right)$$

$$\ln(L(y)) = \ln\left(\frac{1}{2}\right) + \frac{3}{8}y^2$$




Hence, all decision rules will be of the form

$$\rho(y) = \begin{cases} 1 & y^2 > \tau \\ \gamma & y^2 = \tau \\ 0 & y^2 < \tau \end{cases}$$

(we have absorbed the constants $\ln(\frac{1}{2})$ and $3/8$ into $\tau$)

Moreover, since $P(y^2 = \tau) = 0$ for all $\tau \in \mathbb{R}$, we don't have to worry about randomization here.

Hence

$$\rho(y) = \begin{cases} 1 & y^2 \geq \tau \\ 0 & y^2 < \tau \end{cases}$$

will give the same Bayes risk and false positive probability.

a) Bayes decision rule under UCA for general prior

$$\delta^{B\pi} = \begin{cases} 1 & L(y) \geq \frac{\pi_0}{\pi_1} \\ 0 & L(y) < \frac{\pi_0}{\pi_1} \end{cases}$$

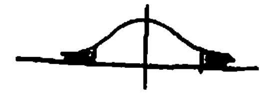
or

$$\boxed{\delta^{B\pi} = \begin{cases} 1 & y^2 \geq \frac{8}{3}\left(\ln\left(\frac{\pi_0}{\pi_1}\right) + \ln(2)\right) \\ 0 & y^2 < 8/3\left(\ln\left(\frac{\pi_0}{\pi_1}\right) + \ln(2)\right) \end{cases}}$$

Bayes – general prior

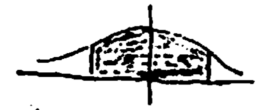
b) Minimax – try the equalizer rule

$$R_0(\delta^{B\pi}) = P\left(y^2 \ge \tfrac{8}{3}\left[\ln\left(\tfrac{\pi_0}{\pi_1}\right) - \ln(2)\right] \,\middle|\, X = x_0\right)$$



$$= 2Q\left(\sqrt{\tfrac{8}{3}\left[\ln(\pi_0/\pi_1) + \ln(2)\right]}\right)$$

$$R_1(\delta^{B\pi}) = P\left(y^2 < \tfrac{8}{3}\left[\ln\left(\tfrac{\pi_0}{\pi_1}\right) - \ln(2)\right] \,\middle|\, X = x_1\right)$$



$$= 1 - 2Q\left(\frac{\sqrt{\tfrac{8}{3}\left[\ln\left(\tfrac{\pi_0}{\pi_1}\right) + \ln(2)\right]}}{2}\right)$$

Set $R_0(\delta^{B\pi}) = R_1(\delta^{B\pi})$

$2Q(x) = 1 - 2Q\left(\frac{x}{2}\right)$ use approximation

$1 - \frac{2x}{\sqrt{2\pi}} = 1 - \left(1 - \frac{x}{\sqrt{2\pi}}\right) \Rightarrow \frac{3x}{\sqrt{2\pi}} = 1 \qquad x = \frac{\sqrt{2\pi}}{3} \approx 0.836$

hence $\sqrt{\tfrac{8}{3}\left[\ln\left(\tfrac{\pi_0}{\pi_1}\right) + \ln(2)\right]} = 0.836$

$$\ln\left(\frac{\pi_0}{\pi_1}\right) + \ln(2) = 0.2618$$

$$\frac{\pi_0}{\pi_1} = 0.6496$$

hence $\left.\begin{array}{l}\pi_0^* = 0.3938\\ \pi_1^* = 0.6062\end{array}\right\}$ <u>least favorable prior</u>

Hence $\delta^{mm} = \delta^{B\pi}$ for $\pi = [0.3938, 0.6062]$

minimax

$$\boxed{\delta^{mm} = \begin{cases} 1 & y^2 \ge .698 \\ 0 & y^2 < .698 \end{cases}}$$

c) Neyman-Pearson

We need to find $v \ge 0$ such that $P(y^2 \ge v^2 \mid X = x_0) = \alpha$.
Since we are dealing with smooth distributions here, we don't need to worry about randomization.

$$P(y^2 \ge v^2 \mid X = x_0) = 2Q(v) = \alpha$$

$$v = Q^{-1}\left(\frac{\alpha}{2}\right)$$

Hence

$$\boxed{\rho^{N-P} = \begin{cases} 1 & y^2 \ge \left(Q^{-1}\left(\frac{\alpha}{2}\right)\right)^2 \\ 0 & y^2 < \left(Q^{-1}\left(\frac{\alpha}{2}\right)\right)^2 \end{cases}}$$

N-P size $\alpha$

3. a) Let's check to see if a UMP decision rule exists

Let $\lambda = a > 1$ (fixed) $\Rightarrow$ Simple binary HT problem

$$L_a(y) = \frac{P_{\lambda=a}(y)}{P_0(y)} = \frac{a e^{-ay}}{e^{-y}} = a e^{-(a-1)y}$$

$$\ln(L_a(y)) = \ln(a) - (a-1)y \gtreqless v$$

Hence, the optimum decision rule for this simple binary problem is

$$\rho^{N-P} = \begin{cases} 1 & y \le \frac{v - \ln(a)}{a-1} \\ 0 & y > \frac{v-\ln(a)}{a-1} \end{cases}$$

$$P_{fp} = P\left(y \le \frac{v-\ln(a)}{a-1} \,\middle|\, X = x_0\right) = \alpha$$

$$= \int_0^{\frac{v-\ln(a)}{a-1}} e^{-y}\,dy = -e^{-y}\Big|_0^{\frac{v-\ln(a)}{a-1}} = 1 - e^{-\left(\frac{v-\ln(a)}{a-1}\right)} = \alpha$$

$$-\ln(1-\alpha) = \frac{v-\ln(a)}{a-1}$$

Hence the critical region is simply defined by $\ln\left(\frac{1}{1-\alpha}\right)$

Not a function of the parameter $a$.

Hence the UMP decision rule is simply

$$\rho^{N-P} = \begin{cases} 1 & y \le \ln\left(\frac{1}{1-\alpha}\right) \\ 0 & y > \ln\left(\frac{1}{1-\alpha}\right) \end{cases}$$

b) power function : $P\left(y \le \ln\left(\frac{1}{1-\alpha}\right) \,\middle|\, X = x_1\right)$

$$= \int_0^{\ln\left(\frac{1}{1-\alpha}\right)} \lambda e^{-\lambda y}\,dy = 1 - (1-\alpha)^{\lambda}$$