ECE531 Spring 2009 Final Examination

Instructions: This exam is worth a total of 500 points. You may consult two double-sided letter-sized sheets of notes (in your own handwriting) and you may use a calculator during the exam. Please show your work on each problem and box/circle your final answers. Points may be deducted for a disorderly presentation of your solution. The exam is closed-book.

1. 200 points. Suppose that an unknown scalar parameter Θ is known to have a prior distribution of

$$p_\Theta(\theta) = \pi(\theta) = \begin{cases} e^{-\theta} & \theta \geq 0 \\ 0 & \theta < 0. \end{cases}$$

   You receive a scalar observation

$$Y = \Theta + U$$

   with U independent of Θ and possessing the distribution

$$p_U(u) = \begin{cases} e^{-u} & u \geq 0 \\ 0 & u < 0. \end{cases}$$

   (a) (50 points) Find the MMSE estimator of the unknown scalar parameter.

   (b) (50 points) Find the LMMSE (linear MMSE) estimator of the unknown scalar parameter. Compare your answer to Part (a).

   (c) (50 points) Find the MAP estimator of the unknown scalar parameter. Hint: Is the MAP estimator unique?

   (d) (50 points) Find the ML estimator of the unknown scalar parameter and compare your answer to Part (c). Hint: The ML estimator is appropriate for non-random parameter estimation, hence your analysis here should not use the prior on Θ.

2. 100 points. Suppose that θ is a scalar parameter that you wish to estimate from N i.i.d. observations received according to the marginal pdf

$$p(y_k\,;\,\theta) \quad = \quad \begin{cases} \exp(\theta - y_k) & y_k \geq \theta \\ 0 & y_k < \theta. \end{cases}$$

   Given $N = 2$ i.i.d. observations, i.e. $y = \{y_0, y_1\}$, find

   (a) (50 points) a *scalar* sufficient statistic $T(y) \in \mathbb{R}$ for the scalar parameter θ and

   (b) (50 points) an MVU estimator for the scalar parameter θ. Given the time limits of the exam, you can assume that your scalar sufficient statistic is complete here.

3. 100 points. Consider the *scalar* dynamical system

$$X[\ell+1] \quad = \quad fX[\ell] + U[\ell] \text{ for } \ell = 0, 1, \ldots$$

where f is a known scalar constant, and the observation model $Y[\ell] = X[\ell] + V[\ell]$ for $\ell = 0, 1, \ldots$. Assume that $\{U[0], U[1], \ldots\}$ and $\{V[0], V[1], \ldots\}$ are independent sequences of i.i.d. $\mathcal{N}(0,1)$ random variables. Also assume that the initial state $X[0] \sim \mathcal{N}(0, \sigma^2)$ and is independent of all $U[\ell]$ and $V[\ell]$.

   (a) (50 points). Determine a value of the initial state variance $\sigma^2 > 0$ such that the Kalman gain is a constant for all $\ell = 0, 1, \ldots$, i.e.,

$$K[\ell] = \Sigma[\ell \,|\, \ell-1] H^\top[\ell] \left(H[\ell]\Sigma[\ell \,|\, \ell-1] H^\top[\ell] + R[\ell] \right)^{-1} \equiv K.$$

   (b) (50 points). Find expressions for the mean-squared prediction error and the mean-squared filtering error for the case derived in Part (a). Comment on the behavior of these errors as $|f| \to 0$ and $|f| \to \infty$.

4. 100 points. Suppose you observe a scalar random variable Y given by

$$Y = N + \theta\lambda$$

where θ is either 0 or 1, λ is a fixed (known) number between 0 and 2, and where N is a random variable that has a uniform density on the interval $(-1, 1)$. We wish to decide between the hypotheses

$$H_0 : \theta = 0$$

versus

$$H_1 : \theta = 1.$$

   (a) (50 points). Find the Neyman-Pearson decision rule for false-alarm probability $\alpha \in [0, 1]$.

   (b) (50 points). Find the power of the Neyman-Pearson decision rule as a function of the false-positive probability and the known parameter λ. Sketch the receiver operating characteristic for the cases $\lambda = 0$, $\lambda = 1$, and $\lambda = 2$.