

ECE531 Homework Assignment Number 1 Solution

1. 4 points. You are given an urn containing one ball, known to be either black or white with equal probability. You drop in a white ball, shake the urn, and remove a white ball. What is the probability that the ball remaining in the urn is white?

Solution: This problem is exactly the same as the coin flipping problem discussed in lecture. After you drop a white ball in the urn, the two unknown states of nature are WW (two white balls in the urn) and WB (one white and one black ball in the urn). The conditional probabilities of removing a white or black ball are then

$$\begin{aligned}\text{Prob}[W | WW] &= 1 \\ \text{Prob}[B | WW] &= 0 \\ \text{Prob}[W | WB] &= 0.5 \\ \text{Prob}[B | WB] &= 0.5\end{aligned}$$

You want to determine the probability that the ball remaining in the urn is white given that you removed a white ball, which is equivalent to the probability that the state of nature is WW given that you removed a white ball, i.e. $\text{Prob}[WW | W]$. You can just use the standard conditional probability expressions given in lecture to compute

$$\text{Prob}[WW | W] = \frac{\text{Prob}[WW, W]}{\text{Prob}[W]} = \frac{\text{Prob}[W | WW]\text{Prob}[WW]}{\text{Prob}[W]}$$

where $\text{Prob}[WW] = 0.5$ is the unconditional probability that the state of nature is WW and $\text{Prob}[W] = \text{Prob}[W | WW]\text{Prob}[WW] + \text{Prob}[W | WB]\text{Prob}[WB] = 1 \cdot 0.5 + 0.5 \cdot 0.5 = 0.75$ is the unconditional probability that you remove a white ball (and where we have used the total probability theorem). Plugging it all in, we get the desired result

$$\text{Prob}[WW | W] = \frac{1 \cdot 0.5}{0.75} = \frac{2}{3}.$$

2. 4 points. A passenger next to you on an airplane (whom you never previously met) tells you she has two children. What is the probability that they are both girls if she says “yes” to
- (a) Is at least one of them a girl?
 - (b) Is the older one a girl?

Solution: The implicit assumption here is that girls and boys are born with equal probability (and that both children can not be born simultaneously, of course). There are a total of four possible states of nature here: GG, GB, BG, BB, where the leftmost letter corresponds to the older of the two children. Each state of nature has the same probability, i.e. 0.25. Hence, for part (a), the probability that both are girls given that at least one is a girl can be computed as

$$\text{Prob}[GG | \text{at least one girl}] = \frac{\text{Prob}[GG, \text{at least one girl}]}{\text{Prob}[\text{at least one girl}]} = \frac{0.25}{0.75} = \frac{1}{3}$$

where $\text{Prob}[GG, \text{at least one girl}] = \text{Prob}[GG] = 0.25$ since the event GG is contained entirely in the event “at least one girl”, and $\text{Prob}[\text{at least one girl}] = \text{Prob}[GG] + \text{Prob}[GB] + \text{Prob}[BG] = 0.75$ since those events are mutually exclusive.

For part (b), the probability that both are girls given that the older one is a girl can be computed as

$$\text{Prob}[GG \mid \text{older one is a girl}] = \frac{\text{Prob}[GG, \text{older one is a girl}]}{\text{Prob}[\text{older one is a girl}]} = \frac{0.25}{0.5} = \frac{1}{2}$$

with similar reasoning as part (a).

3. 4 points. Suppose you have a wired communication system where the transmitter puts +1Vdc on the communication circuit if a binary one is transmitted and puts -1Vdc on the circuit if a binary zero is transmitted. Binary ones are transmitted with probability p and binary zeros are transmitted with probability $1 - p$. The receiver forms its scalar observation by sampling the voltage in the circuit.

The signal at the receiver is corrupted by noise. The sampled voltage at the receiver can be modeled as a random variable $Y = V + W$ where $V \in \{-1, +1\}$ is the voltage applied to the circuit by the transmitter and $W \in \mathbb{R}$ is a Gaussian random variable with zero mean and a standard deviation of 0.3 volts.

- (a) Plot the *posterior* probability that the transmitter sent +1Vdc as a function of the prior probability p given an observation of $Y = 0.1\text{Vdc}$.
- (b) Plot the *posterior* probability that the transmitter sent -1Vdc as a function of the prior probability p given an observation of $Y = 0.1\text{Vdc}$.

Solution: Let A be the event that the transmitter sent +1Vdc (binary one) and let $y = 0.1\text{Vdc}$. In part (a), we want to compute

$$\text{Prob}[A \mid Y = y] = \frac{p_Y(y \mid A)\text{Prob}[A]}{p_Y(y)} = \frac{p_Y(y \mid A) \cdot p}{p_Y(y)}$$

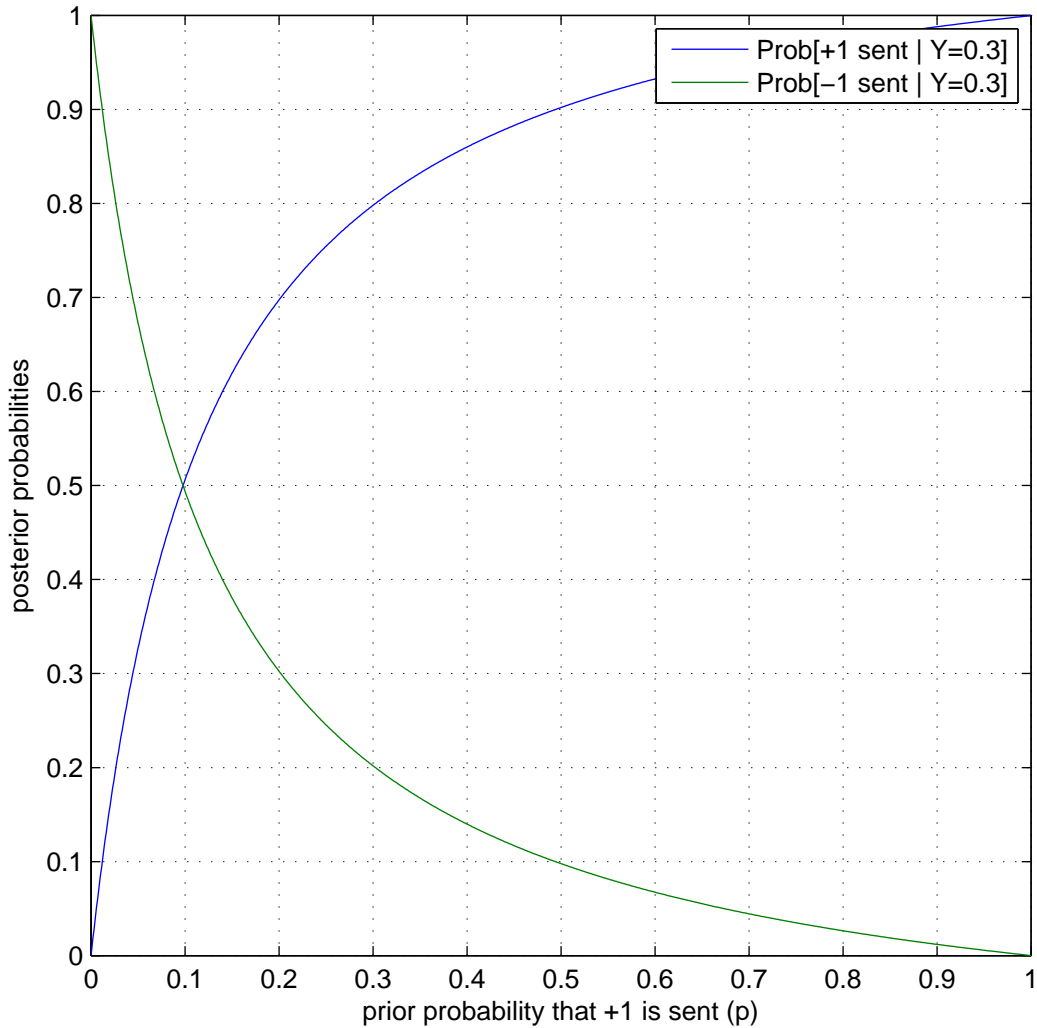
where we have used the appropriate form of Bayes' rule for random variables and events. The conditional pdf $p_Y(y \mid A)$ is simply the pdf of the sampled voltage conditioned on the event that the transmitter sent +1Vdc (binary one). In other words, $Y = 1 + W$ and all that happens here is that Y will have the same pdf as W except with mean of one, i.e.,

$$p_Y(y \mid A) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(y-1)^2}{2\sigma^2}\right\}$$

where $\sigma = 0.3$ is the standard deviation of the noise. The unconditional pdf $p_Y(y)$ can be written as

$$p_Y(y) = p \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(y-1)^2}{2\sigma^2}\right\} + (1-p) \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(y+1)^2}{2\sigma^2}\right\}$$

where we have used the fact that $Y = X + W$ and that the pdf of Y will be equal to the convolution of the pdfs of X and W . So, plugging in $y = 0.1$ and $\sigma = 0.3$, we can generate the following plot (where we have also used the fact that $\text{Pr}[\bar{A} \mid Y = y] = 1 - \text{Pr}[A \mid Y = y]$).



4. 8 points. Two random variables X and Y have the joint probability density function

$$p_{X,Y}(x,y) = \begin{cases} c(x-y)^2 & -1 \leq x \leq 1 \text{ and } -1 \leq y \leq 1 \\ 0 & \text{otherwise.} \end{cases}$$

- (a) Find the appropriate value for c . **Solution:** This can be found by performing the double integral, setting the result equal to one, and solving for c . Skipping the calculus details, the correct answer is $c = \frac{3}{8}$.
- (b) Compute the joint probability $\text{Prob}[X > 0, Y > 0]$. **Solution:** This can be found by performing the double integral over the range $0 < x \leq 1$ and $0 < y \leq 1$ (with the value of c from part (a)). Skipping the calculus details again, the correct answer is $\text{Prob}[X > 0, Y > 0] = \frac{1}{16}$.
- (c) Compute the conditional distribution $p_X(x|y)$. **Solution:** This can be found by using the appropriate form of Bayes' rule, i.e.

$$p_X(x|y) = \frac{p_{X,Y}(x,y)}{p_Y(y)}$$

where the joint probability is given and you just need the marginal. The marginal can

be computed to be

$$p_Y(y) = \frac{3y^2 + 1}{4}$$

hence

$$p_X(x|y) = \frac{3}{2} \cdot \frac{(x-y)^2}{3y^2 + 1}.$$

You should confirm that this is indeed a valid pdf (that it integrates to 1 for any value of y and that it is non-negative).

- (d) Compute the conditional expectation $E[X|y]$. **Solution:** The conditional expectation can be computed from your result in part (c) as

$$E[X|y] = \int_{-1}^1 xp_X(x|y) dx = \frac{-2y}{3y^2 + 1}$$

where y is known (we've conditioned on it). You can see here how knowledge that $Y = y$ affects the mean of X because X and Y are not independent.

- (e) Compute the conditional probability $\text{Prob}[X > 0|Y \leq 0]$. **Solution:** This can be computed as

$$\text{Prob}[X > 0|Y \leq 0] = \frac{\text{Prob}[X > 0, Y \leq 0]}{\text{Prob}[Y \leq 0]} = \frac{7/16}{0.5} = \frac{7}{8}$$

where we know that $\text{Prob}[Y \leq 0] = 0.5$ from the symmetry of the joint pdf.

- (f) Compute the conditional probability $\text{Prob}[X > 0|Y > 0]$. **Solution:** This can be computed easily by using your result from part (b) and the same approach as in part (e). We can write

$$\text{Prob}[X > 0|Y > 0] = \frac{\text{Prob}[X > 0, Y > 0]}{\text{Prob}[Y > 0]} = \frac{1/16}{0.5} = \frac{1}{8}$$

where we know that $\text{Prob}[Y > 0] = 0.5$ from the symmetry of the joint pdf.

- (g) What do parts (e) and (f) imply about the unconditional probability $\text{Prob}[X > 0]$? Explain. **Solution:** The total probability theorem can be used here to write

$$\text{Prob}[X > 0] = \text{Prob}[X > 0|Y \leq 0]\text{Prob}[Y \leq 0] + \text{Prob}[X > 0|Y > 0]\text{Prob}[Y > 0].$$

From the symmetry of the joint pdf, we know that $\text{Prob}[Y \leq 0] = \text{Prob}[Y > 0] = 0.5$, hence

$$\text{Prob}[X > 0] = \frac{7}{8} \cdot \frac{1}{2} + \frac{1}{8} \cdot \frac{1}{2} = \frac{1}{2}$$

as you would expect from the symmetry of the pdf for both X and Y .